

RADIAL LIMITS OF HOLOMORPHIC FUNCTIONS ON THE BALL

A Dissertation

by

MICHAEL C. FULKERSON

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2008

Major Subject: Mathematics

RADIAL LIMITS OF HOLOMORPHIC FUNCTIONS ON THE BALL

A Dissertation

by

MICHAEL C. FULKERSON

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Approved by:

Chair of Committee,	Harold Boas
Committee Members,	Daren Cline
	Roger Smith
	Emil Straube
Head of Department,	Al Boggess

August 2008

Major Subject: Mathematics

ABSTRACT

Radial Limits of Holomorphic Functions on the Ball. (August 2008)

Michael C. Fulkerson, B.S., Central Michigan University;

M.A., Central Michigan University

Chair of Advisory Committee: Dr. Harold Boas

In this dissertation, we consider various aspects of the boundary behavior of holomorphic functions of several complex variables. In dimension one, a characterization of the radial limit zero sets of nonconstant holomorphic functions on the disc has been given by Lusin, Privalov, McMillan, and Berman. In higher dimensions, no such characterization is known for holomorphic functions on the unit ball B . Rudin posed the question as to the existence of nonconstant holomorphic functions on the ball with radial limit zero almost everywhere. Hakim, Sibony, and Dupain showed that such functions exist. Because the characterization in dimension one involves both Lebesgue measure and Baire category, it is natural to also ask whether there exist nonconstant holomorphic functions on the ball having *residual* radial limit zero sets. We show here that such functions exist. We also prove a higher dimensional version of the Lusin-Privalov Radial Uniqueness Theorem, but we show that, in contrast to what is the case in dimension one, the converse does not hold. We show that any characterization of radial limit zero sets on the ball must take into account the “complex structure” on the ball by giving an example that shows that the family of these sets is not closed under orthogonal transformations of the underlying real coordinates. In dimension one, using the theorem of McMillan and Berman, it is easy to see that radial limit zero sets are not closed under unions (even finite unions). Since there is no analogous result in higher dimensions of the McMillan and Berman result, it is

not obvious whether the radial limit zero sets in higher dimensions are closed under finite unions. However, we show that, as is the case in dimension one, these sets are not closed under finite unions. Finally, we show that there are smooth curves of finite length in S that are non-tangential limit uniqueness sets for holomorphic functions on B . This strengthens a result of M. Tsuji.

To my wife, Kimberly, and my daughter, Kaitlyn

ACKNOWLEDGMENTS

I would like to thank Harold Boas for all his help with this research. He has been an invaluable source of knowledge and insight, and I could not have asked for a better advisor and friend. Without his guidance this project would not have been possible.

I would also like to thank Emil Straube, Roger Smith, and Daren Cline for their willingness to be members of my Committee. They are all experts in their respective areas of research, and I appreciate them taking time to help with this project. I also wish to acknowledge Matthew Papanikolas for filling in on such short notice.

Thanks also to my family and friends for their support and encouragement. My wife Kimberly, in particular, has been so understanding of the time and effort it has taken to work my way through graduate school. Her cooperation and understanding have been a source of strength and encouragement to me. And thank you to my little girl, Kaitlyn, who has given me great joy and contentment.

Finally, I would like to thank the people of the mathematics department at Texas A&M University. I have found Texas A&M to not only be a wonderful place to learn mathematics and conduct research, but it has truly been a friendly and pleasant environment as well.

TABLE OF CONTENTS

CHAPTER		Page
I	INTRODUCTION	1
II	BACKGROUND	4
	A. Several Complex Variables Background	4
	B. Boundary Behavior Background	13
	1. Fatou and Lindelöf Theorems	14
	2. Boundary Zero-Sets	20
	3. Miscellaneous Results	23
	C. Baire Category	28
III	THE LUSIN-PRIVALOV THEOREM	31
	A. Preliminaries	31
	B. The Lusin-Privalov Theorem in One Dimension	34
	C. The McMillan-Berman Converse	37
	D. The Lusin-Privalov Theorem in Higher Dimensions	41
	E. Failure of the Converse in Higher Dimensions	44
IV	CONSTRUCTION OF A RESIDUAL RADIAL LIMIT ZERO SET	46
	A. Constructing a Residual Radial Limit Zero Set in One Dimension	47
	1. A New Method in Dimension One	48
	B. Constructing a Residual Radial Limit Zero Set in Higher Dimensions	49
V	OTHER PROPERTIES OF RADIAL LIMIT ZERO SETS	58
	A. Dependence on Complex Structure	58
	B. Not Closed Under Unions	61
	C. Smooth Curves of Finite Length in S that are Non-tangential Uniqueness Sets	62
VI	CONCLUSION	66
	A. Summary of Results	66

CHAPTER	Page
B. Open Questions	67
REFERENCES	69
VITA	73

CHAPTER I

INTRODUCTION

If f is a holomorphic function on the unit disc $D \subset \mathbb{C}$, then it is of interest to know whether f has some kind of realization as a function on the unit circle T . This will, of course, depend on both the function itself and the type of “extension” that we are considering. In the next chapter, we will discuss several different ways in which a function may have a realization on the boundary, but we will focus our attention primarily on *radial limit* extensions.

If f is a real or complex-valued function on D , then f is said to have *radial limit* L at a point $\omega \in T$ if $\lim_{r \rightarrow 1^-} f(r\omega) = L$. We will denote by f^* the radial limit function of f . That is, $f^*(\omega) = \lim_{r \rightarrow 1^-} f(r\omega)$ for each $\omega \in T$ where the limit exists (finitely or infinitely).

A set $E \subset T$ is said to be *metrically dense* in an open arc $\alpha \subset T$ if, for every non-empty open sub-arc $\beta \subset \alpha$, the set $E \cap \beta$ has positive (outer) measure.

The following classical theorem is due to Lusin and Privalov [30].

Theorem 1 (Lusin-Privalov Radial Uniqueness Theorem, 1925). *Let f be a nonconstant holomorphic function on D such that $f^*(\omega) = 0$ for each ω in some set $E \subset T$. Then E satisfies the following property: if α is a non-empty open arc in T , then E is not both metrically dense and second (Baire) category in α .*

J.E. McMillan [31] and R. Berman [7] proved the following full converse of the Lusin-Privalov theorem, thus giving a characterization of the radial limit zero sets of (nonconstant) holomorphic functions on D in terms of measure and category.

The journal model is Transactions of the American Mathematical Society.

Theorem 2 (McMillan, 1966; Berman, 1983). *Suppose a set $E \subset T$ has the following property: if α is a non-empty open arc in T , then E is not both metrically dense and second category in α . Then there exists a nonconstant holomorphic function f on D such that $f^*(\omega) = 0$ for $\omega \in E$.*

Some other classical results in the theory of boundary behavior are Fatou's Theorem, the F. and M. Riesz Theorem, and Lindelöf's Theorem. Fatou's Theorem says that bounded holomorphic functions on D have non-tangential limits almost everywhere on T . The F. and M. Riesz Theorem says that the only bounded holomorphic function on D having radial limit zero on a set of positive measure is the identically zero function. Lindelöf's Theorem says that a bounded holomorphic function on D having limit L along some curve terminating at a point $\omega \in T$ has non-tangential limit L at ω .

In higher dimensions, matters are more complicated. There are many useful tools in dimension one that are unavailable in higher dimensions (e.g., the Riemann Mapping Theorem, Mergelyan's Theorem, etc.). However, there are some theorems (e.g. Fatou's Theorem) that are, in some ways, "better" in higher dimensions. There is a type of convergence in higher dimensions, called admissible convergence, which allows for parabolic approach to the boundary in complex tangential directions that is more suitable, in some sense, than non-tangential convergence for functions of several complex variables. The approach regions (called Korányi regions) that are associated with this type of convergence arise from using a Hardy-Littlewood maximal function with respect to a nonisotropic metric on S .

In this dissertation, we will consider various questions related to the boundary behavior of holomorphic functions on the unit ball in \mathbb{C}^n . In Chapter II, we will provide some background information on several complex variables. We will state

the relevant definitions and will prove some results pertaining to: holomorphic and pluriharmonic functions, Hardy spaces, the Green's function for the Laplacian, the Poisson kernel, etc. We will also state (but not prove) some important theorems about the boundary behavior of holomorphic and harmonic functions. This will include background information on non-tangential limits and K-limits (i.e., admissible convergence).

In Chapter III, we will state and prove the classical Lusin-Privalov Radial Uniqueness Theorem. We will also give Berman's proof of the converse. We will prove a higher dimensional analogue of the Lusin-Privalov Theorem, but we will give an example to show that the converse does not hold in \mathbb{C}^n ($n \geq 2$).

In Chapter IV, we will use a new method to construct a radial limit zero subset of the unit circle T that is *residual* in T . The method that Privalov used (see [33]) cannot be extended to higher dimensions. We show how the new method can be generalized to construct a nonconstant holomorphic function on B with radial limit zero on a residual subset of S .

In Chapter V, we will show that the radial limit zero sets depend on the “complex structure” on S . By this we mean that there is a set $E \subset S$, an orthogonal transformation \mathcal{O} of the underlying real coordinates, and a nonconstant holomorphic function on B with radial limit zero on E such that there does *not* exist a nonconstant holomorphic function with radial limit zero on $\mathcal{O}(E)$. We will also show in Chapter V that the family of radial limit zero sets is not closed under unions (even finite unions).

In Chapter VI, we provide a summary of the results proved in this dissertation. We also state some questions that are still open. Finally, we discuss some possible future directions for research.

CHAPTER II

BACKGROUND

In the first section of this chapter, we will give some basic background information about several complex variables. We will define what it means for a function of several variables to be holomorphic, and we will state and prove various properties of holomorphic functions. In the second section, some background information will be given with regard to various aspects of the boundary behavior of holomorphic and harmonic functions. We will state a broad array of results, but we will refer the reader to the relevant papers for the proofs. In the third section, we will give a brief review of some material related to the Baire Category Theorem.

A. Several Complex Variables Background

To each point $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ we may associate the point $(x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n}$ where $z_j = x_j + iy_j$ for each $j = 1, \dots, n$. In the usual way, we define an inner product on \mathbb{C}^n as follows: $\langle z, w \rangle = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n$. Using this inner product, we define a norm on \mathbb{C}^n in the standard fashion: $\|z\| = \langle z, z \rangle^{1/2} = (|z_1|^2 + \dots + |z_n|^2)^{1/2} = (|x_1|^2 + |y_1|^2 + \dots + |x_n|^2 + |y_n|^2)^{1/2}$. (We will usually just use the notation $|z|$ instead of $\|z\|$.)

For $w \in \mathbb{C}^n$ and $r > 0$ define

$$B(w, r) = \{z \in \mathbb{C}^n : |z - w| < r\}$$

and

$$S(w, r) = \{z \in \mathbb{C}^n : |z - w| = r\}.$$

So $B(w, r)$ is the ball centered at w with radius r , and $S(w, r)$ is the boundary of

$B(w, r)$. For $n = 1$, we will use D and T to denote $B(0, 1)$ and $S(0, 1)$, respectively, and for $n \geq 2$, we will use B and S to denote $B(0, 1)$ and $S(0, 1)$, respectively.

We also define

$$D^n(w, r) = \{z \in \mathbb{C}^n : |z_j - w_j| < r, j = 1, \dots, n\}$$

and

$$T^n(w, r) = \{z \in \mathbb{C}^n : |z_j - w_j| = r, j = 1, \dots, n\}.$$

So $D^n(w, r)$ is a *polydisc* centered at w with radius r . And $T^n(w, r)$ is a *torus* centered at w with radius r . Note that $T^n(w, r)$ is not the whole boundary of $D^n(w, r)$, but only part of it. We will use D^n and T^n to denote $D^n(0, 1)$ and $T^n(0, 1)$, respectively.

Let Ω be a *domain* in \mathbb{C}^n . By this we mean that Ω is open and connected. A function $f : \Omega \rightarrow \mathbb{C}$ is said to be *holomorphic* if it is holomorphic in each variable separately. Another way of stating this condition is that f satisfies the Cauchy-Reimann equations in each variable separately. That is,

$$\frac{\partial f}{\partial \bar{z}_j} = 0$$

for each $j = 1, \dots, n$. Here

$$\frac{\partial}{\partial \bar{z}_j} \equiv \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$$

where $z_j = x_j + iy_j$. It will also be useful to define

$$\frac{\partial}{\partial z_j} \equiv \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right)$$

for each $j = 1, \dots, n$. It is a non-trivial result of Hartogs that holomorphic functions are continuous in all variables jointly.

Let $\Omega \subset \mathbb{C}^n$ be a domain that contains $\overline{D^n}$, the closure of the unit polydisc. If

$f : \Omega \rightarrow \mathbb{C}$ is holomorphic, then

$$f(z) = \int_{T^n} f(\omega) \prod_{j=1}^n \frac{1}{1 - \bar{\omega}_j z_j} d\lambda_n(\omega),$$

where λ_n is Lebesgue measure divided by $(2\pi)^n$ so that $\lambda_n(T^n) = 1$. This is known as the *Cauchy formula* in D^n . This formula is easily derived from the familiar Cauchy Integral Formula in \mathbb{C} : If f is holomorphic in a neighborhood of D and $z \in D$, then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_T \frac{f(w)}{w - z} dw \\ &= \frac{1}{2\pi i} \int_T \frac{f(w)}{w - z} 2\pi i w d\lambda_1(w) \\ &= \int_T \frac{f(w)}{1 - \bar{w}z} d\lambda_1(w). \end{aligned}$$

The Cauchy formula in D^n can be obtained from this formula by repeated integration in the individual variables. Note that, since $|\bar{w}_j z_j| < 1$ for each $j = 1, \dots, n$, we may write

$$\begin{aligned} \prod_{j=1}^n \frac{1}{1 - \bar{w}_j z_j} &= \left(\sum_{a_1=1}^{\infty} (\bar{w}_1 z_1)^{a_1} \right) \cdot \dots \cdot \left(\sum_{a_n=1}^{\infty} (\bar{w}_n z_n)^{a_n} \right) \\ &= \sum_{\alpha} \bar{w}^{\alpha} z^{\alpha}, \end{aligned}$$

where α ranges over all multi-indices. This series converges uniformly on compact subsets of D^n . The Cauchy formula may then be written

$$\begin{aligned} f(z) &= \int_{T^n} f(w) \left(\sum_{\alpha} \bar{w}^{\alpha} z^{\alpha} \right) d\lambda_n(w) \\ &= \sum_{\alpha} \left(\int_{T^n} f(w) \bar{w}^{\alpha} d\lambda_n(w) \right) z^{\alpha} \\ &= \sum_{\alpha} c_{\alpha} z^{\alpha}. \end{aligned}$$

Thus f has a power series expansion in D^n , where the coefficients may be determined

by the above formula. The power series converges absolutely and uniformly on compact subsets of D^n , so the order of summation does not matter. There is certainly nothing special about expanding about the origin or about the radius being 1. So holomorphic functions have local power series representations.

A *complex line* in \mathbb{C}^n is a set of the form $\{a + b\omega : \omega \in \mathbb{C}\}$ where $a, b \in \mathbb{C}^n$. It turns out that holomorphic functions must be holomorphic on all complex lines (not just those complex lines that are parallel to the coordinate axes). To see this, let f be holomorphic on a domain $\Omega \subset \mathbb{C}^n$, and fix $a \in \Omega$ and $b \in \mathbb{C}^n$. Suppose U is a neighborhood of 0 in \mathbb{C} such that for each $w \in U$, $a + wb \in \Omega$. We claim that the function $g : U \rightarrow \mathbb{C}$ defined by $g(w) = f(a + wb)$ is holomorphic. To see this, note that f may be expanded as

$$f(z) = \sum_{\alpha} c_{\alpha} z^{\alpha}.$$

We then have

$$\begin{aligned} g(w) &= f(a + wb) \\ &= \sum_{\alpha} c_{\alpha} (a + wb)^{\alpha} \\ &= \sum_{\alpha} c_{\alpha} (a_1 + wb_1)^{\alpha_1} \cdot \dots \cdot (a_n + wb_n)^{\alpha_n} \end{aligned}$$

This sum (of holomorphic functions) converges uniformly on compact subsets of U , so g is holomorphic on U .

Let Ω be a domain in \mathbb{C}^n . If u is a class C^2 function on Ω , we define the *Laplacian* of u as follows:

$$\Delta u = \sum_{j=1}^n \left(\frac{\partial^2 u}{\partial x_j^2} + \frac{\partial^2 u}{\partial y_j^2} \right).$$

It is easy to check that

$$\Delta u = 4 \sum_{j=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}.$$

A C^2 function u on a domain Ω is said to be *harmonic* if $\Delta u \equiv 0$ on Ω . It is a standard fact of complex analysis in the plane that the real part of a holomorphic function is harmonic and that (real-valued) harmonic functions are locally real parts of holomorphic functions.

Since holomorphic functions in \mathbb{C}^n must be holomorphic on complex lines, the real part of a holomorphic function of several variables must be harmonic on complex lines. Functions with this property are called *pluriharmonic*. That is, a (real or complex-valued) function u on a domain $\Omega \subset \mathbb{C}^n$ is pluriharmonic if, for each $a \in \Omega$ and $b \in \mathbb{C}^n$, the function $w \mapsto u(a + wb)$ is harmonic in some neighborhood of $0 \in \mathbb{C}$. It is not difficult to see that all pluriharmonic functions are harmonic, but not conversely. We also point out the interesting fact that there are functions which are harmonic in each variable separately but are not pluriharmonic (for example, $u(z_1, z_2) = \operatorname{Re}(z_1 \bar{z}_2)$).

We now state the chain rule. Let Ω be open in \mathbb{C}^k . Suppose $F = (f_1, \dots, f_n) : \Omega \rightarrow \mathbb{C}^n$ and $g : F(\Omega) \rightarrow \mathbb{C}$ where f_1, \dots, f_n, g are all C^1 . Let $h = g \circ F$. Then for each $j = 1, \dots, k$ and for fixed $z \in \Omega$, we have

$$\frac{\partial h}{\partial z_j}(z) = \sum_{l=1}^n \left(\frac{\partial g}{\partial z_l}(F(z)) \cdot \frac{\partial f_l}{\partial z_j}(z) + \frac{\partial g}{\partial \bar{z}_l}(F(z)) \cdot \frac{\partial \bar{f}_l}{\partial z_j}(z) \right)$$

and

$$\frac{\partial h}{\partial \bar{z}_j}(z) = \sum_{l=1}^n \left(\frac{\partial g}{\partial z_l}(F(z)) \cdot \frac{\partial f_l}{\partial \bar{z}_j}(z) + \frac{\partial g}{\partial \bar{z}_l}(F(z)) \cdot \frac{\partial \bar{f}_l}{\partial \bar{z}_j}(z) \right).$$

These formulas may be obtained from the ordinary chain rule in real variables. The computation is elementary, yet somewhat tedious.

We will now use the chain rule to see that a C^2 function $u : \Omega \rightarrow \mathbb{R}$ is pluriharmonic if and only if $\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \equiv 0$ for each $j, k = 1, \dots, n$. Let $a \in \Omega$ and $b \in \mathbb{C}^n$. Let $V \subset \mathbb{C}$ be a neighborhood of 0 such that $\{a + zb : z \in V\} \subset \Omega$. Define

$g_{a,b}(z) = u(a + zb)$. Then

$$\begin{aligned}
\Delta g_{a,b}(z) &= 4 \frac{\partial}{\partial z} \left(\frac{\partial g_{a,b}}{\partial \bar{z}}(z) \right) \\
&= 4 \frac{\partial}{\partial z} \left(\sum_{k=1}^n \left(\frac{\partial u}{\partial z_k}(a + zb) \cdot \frac{\partial(a_k + zb_k)}{\partial \bar{z}}(z) + \frac{\partial u}{\partial \bar{z}_k}(a + zb) \cdot \frac{\partial \overline{(a_k + zb_k)}}{\partial \bar{z}}(z) \right) \right) \\
&= 4 \frac{\partial}{\partial z} \left(\sum_{k=1}^n \left(\frac{\partial u}{\partial z_k}(a + zb) \cdot 0 + \frac{\partial u}{\partial \bar{z}_k}(a + zb) \cdot \bar{b}_k \right) \right) \\
&= 4 \sum_{k=1}^n \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial \bar{z}_k}(a + zb) \cdot \bar{b}_k \right) \\
&= 4 \sum_{k=1}^n \bar{b}_k \sum_{j=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a + zb) \cdot b_j + \frac{\partial^2 u}{\partial \bar{z}_j \partial \bar{z}_k}(a + zb) \cdot 0 \\
&= 4 \sum_{k=1}^n \sum_{j=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a + zb) \cdot \bar{b}_k \cdot b_j.
\end{aligned}$$

So

$$\Delta g_{a,b}(0) = 4 \sum_{k=1}^n \sum_{j=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a) \cdot \bar{b}_k \cdot b_j.$$

Now suppose u is pluriharmonic. This means that u is harmonic on complex lines. But this is true if and only if $\Delta g_{a,b}(0) = 0$ for *every* $a \in \Omega$ and $b \in \mathbb{C}^n$. By the formula for $\Delta g_{a,b}(0)$ above, the only way this can happen is if $\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \equiv 0$ for each $j, k = 1, \dots, n$. Conversely, if $\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \equiv 0$ for each $j, k = 1, \dots, n$, then the formula above gives that $\Delta g_{a,b}(0) = 0$ for every $a \in \Omega$ and $b \in \mathbb{C}^n$. We conclude that u is pluriharmonic if and only if $\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \equiv 0$ for each $j, k = 1, \dots, n$.

Let σ be Lebesgue measure on S , normalized so that $\sigma(S) = 1$. Let $\Omega \subset \mathbb{C}^n$ be a domain. An upper semicontinuous function $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ is said to be *subharmonic* if, for each $a \in \Omega$ and each $r > 0$ such that $\overline{B(a, r)} \subset \Omega$,

$$u(a) \leq \int_S u(r + az) d\sigma(z).$$

The property is actually a local one. That is, only sufficiently small $r > 0$ need to be

checked.

We now show that if $f : \Omega \rightarrow \mathbb{C}$ is holomorphic, then $u := \log |f|$ is subharmonic. We prove the $n = 1$ case first. The property is trivially satisfied at any point $a \in \Omega$ such that $f(a) = 0$. So suppose $a \in \Omega$ is not a zero of f . Let $r > 0$ be so small that $\overline{B(a, r)}$ does not contain any zeros of f . This is possible since the zeros are isolated. Then $\log |f|$ is the real part of the holomorphic function $\log f$ on $B(a, r)$, so $\log |f|$ is harmonic (hence also subharmonic) on $B(a, r)$. We now prove the $n \geq 2$ case. Let $a \in \Omega$ and $z \in S$. By the result for $n = 1$, we have that for all $r > 0$ satisfying $\overline{B(a, r)} \subset \Omega$,

$$u(a) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + re^{i\theta}z) d\theta.$$

By the rotational invariance of σ , we have that for fixed $\psi \in (-\pi, \pi]$,

$$\int_S u(a + rz) d\sigma(z) = \int_S u(a + re^{i\psi}z) d\sigma(z).$$

We thus have

$$\begin{aligned} \int_S u(a + rz) d\sigma(z) &= \int_S u(a + re^{i\psi}z) d\sigma(z) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\int_S u(a + re^{i\theta}z) d\sigma(z) \right) d\theta \\ &= \int_S \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + re^{i\theta}z) d\theta \right) d\sigma(z) \\ &\geq \int_S u(a) d\sigma(z) \\ &= u(a). \end{aligned}$$

In the third equality, we used Fubini's Theorem. We have thus proved that if f is holomorphic on $\Omega \subset \mathbb{C}^n$, then $\log |f|$ is subharmonic.

We will show in Proposition 4 that real-valued pluriharmonic functions are locally real parts of holomorphic functions. But we first need the following lemma.

Lemma 3 (Poincaré Lemma). *Let $\alpha = \sum_{j=1}^n \alpha_j dx_j$ be a differential form on \mathbb{R}^n , where each α_j is a C^1 function on \mathbb{R}^n . If $d\alpha = 0$ on a convex open set Ω , then there is a function g on Ω such that $dg = \alpha$. Moreover, if the α_j 's are real, then g can be taken to be real.*

For a proof of the Poincaré Lemma, see [10, pp. 288-289] or [26, pp. 92-93].

Proposition 4. *If u is a (real-valued) pluriharmonic function on a ball, then there is a holomorphic function f on the ball such that $u = \operatorname{Re} f$.*

Proof. Let $\alpha = i(\bar{\partial}u - \partial u)$. Note that α is real and satisfies $d\alpha = i(\partial + \bar{\partial})(\bar{\partial}u - \partial u) = 0$, since u is pluriharmonic (i.e., $\partial\bar{\partial}u = 0$). That is, α is a closed form. By the Poincaré Lemma, there is a real function v such that $dv = \alpha$. Thus $d(iv) = i\alpha = \partial u - \bar{\partial}u$. But also $d(iv) = \partial(iv) + \bar{\partial}(iv)$. Thus $\bar{\partial}(iv) = -\bar{\partial}u$. This gives $\bar{\partial}(u + iv) = 0$. Let $f = u + iv$. □

We now state Green's Theorem, which is a consequence of the Divergence Theorem.

Theorem 5 (Green's Theorem). *Let Ω be a C^2 domain in \mathbb{R}^n , and let ν denote the outward unit normal vector field on $\partial\Omega$. If u and v are C^2 functions on $\bar{\Omega}$, then*

$$\int_{\partial\Omega} (\nu v)u - (\nu u)v d\sigma = \int_{\Omega} (\Delta v)u - (\Delta u)v dV$$

where $d\sigma$ is Lebesgue (area) measure on $\partial\Omega$ and where dV is Lebesgue (volume) measure on Ω .

For $n \in \mathbb{N}$ ($n \geq 2$), define

$$\Gamma_n(x) = \begin{cases} \frac{1}{2\pi} \log |x| & \text{if } n = 2 \\ \frac{-1}{(n-2) \cdot \omega_{n-1} \cdot |x|^{n-2}} & \text{if } n \geq 3 \end{cases}$$

where ω_{n-1} is the area measure of the $(n-1)$ -dimensional unit sphere in \mathbb{R}^n . This area measure is easily computed (see [2, pp. 239-240]) to be:

$$\omega_{n-1} = \begin{cases} \frac{n \cdot \pi^{n/2}}{(n/2)!} & \text{if } n \text{ is even,} \\ \frac{n \cdot 2^{(n+1)/2} \pi^{(n-1)/2}}{1 \cdot 3 \cdot 5 \cdots n} & \text{if } n \text{ is odd.} \end{cases}$$

If ϕ is a compactly supported C^∞ function on \mathbb{R}^n , then

$$\int_{\mathbb{R}^n} (\Delta \phi) \Gamma_n dV = \phi(0).$$

Thus, in the sense of distributions, $\Delta \Gamma_n = \delta$. So Γ_n is called the *fundamental solution for the Laplacian* in \mathbb{R}^n . It can be checked that $\Delta(\Gamma_n * \phi) = \phi$.

Let Ω be a bounded domain in \mathbb{R}^n on which the Dirichlet problem can be solved. For $x \in \Omega$, define a function h_x on $\partial\Omega$ by

$$h_x(y) = \Gamma_n(y - x)|_{y \in \partial\Omega}.$$

Let $H(x, y)$ be the solution of the Dirichlet problem on Ω with boundary data $h_x(y)$. Define the *Green's function* for Ω to be $G_\Omega(x, y) = H(x, y) - \Gamma_n(y - x)$. Note that for fixed $x \in \Omega$, G_Ω is harmonic (in y) for $y \in \Omega \setminus \{x\}$ and is superharmonic (in y) for all $y \in \Omega$. Also, $G_\Omega(x, y)|_{y \in \partial\Omega} = 0$ for fixed $x \in \Omega$.

Let $G_\Omega(x, y)$ be the Green's function for some bounded domain $\Omega \subset \mathbb{R}^n$ with C^2 boundary. Define the *Poisson kernel* $P_\Omega(x, y) : \Omega \times \partial\Omega \rightarrow \mathbb{R}$ by

$$P_\Omega(x, y) = -\nu_y G(x, y)$$

where ν is the outward unit normal vector field on Ω . If u is continuous on $\overline{\Omega}$ and harmonic on Ω , then for each $x \in \Omega$,

$$u(x) = \int_{\partial\Omega} P_\Omega(x, y) u(y) d\sigma(y),$$

where $d\sigma$ is Lebesgue measure on Ω .

Note that $G(x, y) \geq 0$ because, for fixed $x \in \Omega$, it is superharmonic in y and identically zero on the boundary. Using the Hopf Lemma, since $G(x, y)$ is harmonic (in y) away from x and using the fact that for fixed x each point of the boundary of Ω is a minimum of $G(x, y)$, we conclude that $\nu_y G(x, y) < 0$. Thus $P(x, y) > 0$. We also have that $P(x, y)$ is harmonic in x for each fixed $y \in \partial\Omega$, and that

$$\|P(x, \cdot)\|_{L^1(\partial\Omega, d\lambda)} = 1.$$

The Poisson kernel may also be defined on a domain with non-smooth boundary, as long as the boundary is at least “piecewise” C^2 .

B. Boundary Behavior Background

Define $u : D \rightarrow \mathbb{R}$ by

$$u = \operatorname{Im} \left(\left(\frac{1+z}{1-z} \right)^2 \right).$$

We claim that $u^*(z) = 0$ for each $z \in T$. To show this, we consider the mapping properties of

$$h(z) := \left(\frac{1+z}{1-z} \right)^2.$$

The function $(1+z)/(1-z)$, called the *Cayley transform*, is a biholomorphic mapping of D onto the right half-plane. So h is a biholomorphic mapping of D onto the set

$$\mathbb{C} \setminus \{\mathbb{R}^- \cup \{0\}\}.$$

For fixed $\theta \in (0, 2\pi)$, h maps the curve $\gamma(t) = te^{i\theta}$, $0 < t < 1$, to a curve in $\mathbb{C} \setminus \{\mathbb{R}^- \cup \{0\}\}$ that begins at $1 \in \mathbb{C}$ and ends at a point of the negative real axis. Thus $u^*(e^{i\theta}) = 0$. Also, since $h(r)$ is real for $0 < r < 1$, we have $u^*(1) = 0$. It should be noted that the function u has *general* limit zero at each point of $T \setminus \{1\}$, but at 1

it only has *radial* limit zero.

Let $\omega \in S$ and let $t \in (1, \infty)$. The *non-tangential approach region* (or *Stolz region*) with vertex ω and aperture t is defined to be

$$\Gamma_t(\omega) = \{z \in B : |z - \omega| < t(1 - |z|)\}.$$

A function f on B is said to have *non-tangential limit* L at a point $\omega \in S$ if, for every $t \in (1, \infty)$,

$$\lim_{\Gamma_t(\omega) \ni z \rightarrow \omega} f(z) = L.$$

It is possible for a holomorphic function on B to have non-tangential limit at a point $\omega \in S$ without having *general* limit at ω . For example, it can be shown that the holomorphic function $h : D \rightarrow \mathbb{C}$ defined by

$$h(z) = e^{\frac{z+1}{z-1}}$$

has non-tangential limit zero at the point $1 \in T$ but it does not have a general limit at 1.

1. Fatou and Lindelöf Theorems

For *bounded* holomorphic functions, we have the following classical result of Fatou [15].

Theorem 6 (Fatou's Theorem, 1906). *A bounded holomorphic function on D has radial limits almost everywhere on T .*

Can the conditions of *boundedness* and *holomorphicity* be relaxed in the above theorem? We will later see that the conditions can be relaxed somewhat, but they cannot be completely eliminated. The function

$$f(z) = \sin \left(\frac{1}{1 - |z|^2} \right)$$

clearly does not have a radial limit at *any* point of T . Note that f is bounded and even real analytic, but not holomorphic. Also, theorem of Bagemihl and Seidel [3] can be used to construct an holomorphic function (which is necessarily unbounded) such that the set of points of T for which the radial limit exists has measure zero. In fact, Runge's Theorem can be used to construct a holomorphic function on D having radial limits at *no* point of T .

We now state Lindelöf's Theorem [29]. We will give a proof in Chapter III.

Theorem 7 (Lindelöf, 1915). *Suppose f is a bounded holomorphic function on D and $\gamma : [0, 1) \rightarrow D$ is a continuous curve such that $\gamma(t) \rightarrow \omega \in T$ as $t \rightarrow 1$. If*

$$\lim_{t \rightarrow 1} f(\gamma(t)) = L$$

exists, then f has non-tangential limit L at ω .

Combining Fatou's Theorem with Lindelöf's Theorem, we obtain the following corollary.

Corollary 8. *A bounded holomorphic function on D has non-tangential limits almost everywhere on T .*

It is important to remember the *boundedness* condition in each of the above results. In fact, none of the stated results are true for general unbounded functions. Counterexamples can be easily obtained using a theorem (to be stated later) of Bagemihl and Seidel [3].

It should be noted, however, that the boundedness condition on f in Fatou's Theorem and in Corollary 8 can indeed be replaced by various weaker conditions. To state these results we will need a few definitions.

For $0 < p < \infty$, we define the *Hardy Spaces* (in \mathbb{C}):

$$H^p(D) = \left\{ f \text{ holo. on } D : \sup_{0 < r < 1} \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right]^{1/p} < \infty \right\}.$$

Also:

$$H^\infty(D) = \left\{ f \text{ holo. on } D : \sup_D |f| < \infty \right\}.$$

We define the *Hardy Spaces* $H^p(B)$ (in \mathbb{C}^n) in a similar fashion.

For $x > 0$, let $\log^+ x = \max(\log x, 0)$, and let $\log^+ 0 = 0$. A holomorphic function f on D is said to be in the *Nevanlinna class* $N(D)$ if

$$\sup_{0 < r < 1} \int_T \log^+ |f(rz)| d\lambda < \infty.$$

Similarly, we may define in higher dimensions the *Nevanlinna class* $N(B)$. Note that the Nevanlinna class contains all of the Hardy Spaces. That is, $H^p(B) \subset N(B)$ for all $0 < p \leq \infty$.

The following theorem, whose proof can be found in [13, p. 277], is a generalization of Corollary 8.

Theorem 9. *If $f \in N(D)$, then f has non-tangential limits almost everywhere on T .*

Since $N(D) \supset H^p(D)$ (for $p \in (0, +\infty]$), the above theorem is also true for each of the Hardy Spaces.

In \mathbb{C}^n the situation is even better, but we first need some definitions. For $t > 1$ and $\omega \in S$, we define:

$$D_t(\omega) = \{z \in B : |1 - \langle z, \omega \rangle| < \frac{t}{2}(1 - |z|^2)\}$$

Note that for every fixed $\omega \in S$, the regions $D_t(\omega)$ fill up B as $t \rightarrow \infty$. It should also be noted that the regions $D_t(\omega)$ are not restricted to be non-tangential in certain

directions (namely, the complex-tangential directions). We say that a function $f : B \rightarrow \mathbb{C}$ has *K-limit* L (or *admissible limit* L) at $\omega \in S$, and we write

$$(K\text{-}\lim f)(\omega) = L$$

if the following is true: For every $t > 1$ and for every sequence $\{z_j\}$ in $D_t(\omega) \cap B$ that converges to ω , we have $f(z_j) \rightarrow L$ as $j \rightarrow \infty$.

A. Korányi [25] proved the following:

Theorem 10 (Korányi, 1969). *If $f \in N(B)$, then f has K -limits almost everywhere on S .*

In particular, such a function has non-tangential limits almost everywhere.

Recall that for bounded holomorphic functions on D , the existence of a radial limit at a point of T implies the existence of a non-tangential limit at that point (this is Lindelöf's Theorem). However, if f is only assumed to be in $N(D)$ or in $H^p(D)$ (for $0 < p < \infty$), then the existence of a radial limit at a point of T does *not* necessarily imply the existence of a non-tangential limit.

It is also natural to ask whether Lindelöf's Theorem has an analogue in higher dimensions. It does, but we will first need a few definitions.

A ζ -*curve* is a continuous map $\Gamma : [0, 1) \rightarrow B$ such that $\Gamma(t) \rightarrow \zeta$ as $t \rightarrow 1$. If Γ is a ζ -curve, the orthogonal projection of Γ into the complex line through 0 and ζ will be denoted by γ . That is,

$$\gamma = \langle \Gamma, \zeta \rangle \zeta.$$

It is not hard to see that

$$\frac{|\Gamma - \gamma|^2}{1 - |\gamma|^2} < 1.$$

A ζ -curve Γ is called *special* if

$$\lim_{t \rightarrow 1} \frac{\Gamma(t) - \gamma(t)}{1 - |\gamma(t)|^2} = 0$$

and is called *restricted* if it is both special and satisfies (for $0 \leq t < 1$):

$$\frac{|\gamma(t) - \zeta|}{1 - |\gamma(t)|} \leq A$$

for some $A < \infty$. Note that a special ζ -curve Γ is restricted if and only if its projection γ is non-tangential.

A function $f : B \rightarrow \mathbb{C}$ is said to have *restricted K -limit* L at ζ if

$$\lim_{t \rightarrow 1} f(\Gamma(t)) = L$$

for every restricted ζ -curve Γ . It is possible for a holomorphic function $f : B \rightarrow \mathbb{C}$ to have a restricted K -limit at a point $\omega \in S$ without having a K -limit at ω (but not conversely). It is even possible if we assume that f is bounded. Note also that having a restricted K -limit at a point is stronger than having a non-tangential limit. Thus K -limits are the “best,” restricted K -limits are second best, and non-tangential limits are the weakest of the three.

In 1973, Čirka [11] proved the following higher-dimensional analogue of Lindelöf’s Theorem:

Theorem 11 (Čirka, 1973). *Suppose f is a bounded holomorphic function on B , $\zeta \in S$, Γ_0 is a special ζ -curve, and*

$$\lim_{t \rightarrow 1} f(\Gamma_0(t)) = L.$$

Then f has restricted K -limit L at ζ .

The Čirka theorem is stronger than the Lindelöf theorem in that the conclusion

gives a convergence that is stronger than non-tangential. But it is weaker in that the hypothesis requires the approach curve to be a special ζ -curve rather than an arbitrary curve.

It would be nice if we had a Lindelöf-type theorem that guaranteed K -limits rather than restricted K -limits. Recently, Krantz [28] proved the following:

Theorem 12 (Krantz, 2007). *Let f be a bounded holomorphic function on the unit ball $B \subset \mathbb{C}^2$. Let*

$$M = \{(s + i0, t + i0) : s, t \in \mathbb{R}, 0 < s < 1, 0 < |t| < \sqrt{2 - 2s}\}.$$

Suppose that $\rho : T \rightarrow \mathbb{R}^2$ is a C^2 function with bounded first and second derivatives, such that (writing $\rho(s, t) = (\rho_1(s, t), \rho_2(s, t))$)

$$\mathcal{M} = \{(s + i\rho_1(s, t), t + i\rho_2(s, t)) : (s, t) \in M\}$$

is a two-dimensional, totally real manifold in B . Let $\mathbf{1} = (1 + 0i, 0 + 0i)$. Suppose that

$$\lim_{\mathcal{M} \ni z \rightarrow \mathbf{1}} f(z) = \lambda \in \mathbb{C}$$

exists. Then, for any $\alpha > 1$,

$$\lim_{D_\alpha(\mathbf{1}) \ni z \rightarrow \mathbf{1}} f(z) = \lambda.$$

Basically the Krantz theorem says that if we assume the existence of a limit along a somewhat “arbitrary” 2-dimensional totally real manifold terminating at a point of the boundary, then we have a K -limit at that point.

2. Boundary Zero-Sets

We now shift our focus a bit by discussing results related to the types of sets where a holomorphic function can have radial limit *zero*. The following classical result is due to F. and M. Riesz [34].

Theorem 13 (F. and M. Riesz, 1916). *If there exists a nonconstant bounded holomorphic function on D with $f^*(\omega) = 0$ for $\omega \in E \subset T$, then E has measure zero.*

Privalov [33] showed that the converse of Theorem 13 is also true.

Theorem 14 (Privalov, 1956). *Let $E \subset T$ be a set of measure zero. Then there exists a nonconstant bounded holomorphic function on D with radial limit zero on E .*

Compare the following theorem (a proof of which can be found in [13]) to the F. and M. Riesz Theorem.

Theorem 15. *If f is a meromorphic function on D with non-tangential limit zero on a set of positive measure in T , then $f \equiv 0$.*

A subset E of T is said to be *metrically dense* in an open arc α if, for every non-empty subarc β of α , the set $E \cap \beta$ has positive measure.

The following is the Lusin-Privalov radial uniqueness theorem [30].

Theorem 16 (Lusin-Privalov Radial Uniqueness Theorem, 1925). *Let f be a nonconstant holomorphic function on D such that $f^*(\omega) = 0$ for each ω in some set $E \subset T$. Then E satisfies the following property: if α is a non-empty open arc in T , then E is not both metrically dense and of second category in α .*

J.E. McMillan [31] and R. Berman [7] proved the full converse of the Lusin-Privalov theorem.

Theorem 17 (McMillan, 1966; Berman, 1983). *Suppose a subset E of T has the following property: if α is a non-empty open arc in T , then E is not both metrically dense and of second category in α . Then there exists a nonconstant holomorphic function f on D such that $f^*(\omega) = 0$ for $\omega \in E$.*

The function f in the above theorem can be taken to be nowhere zero on D . The following important theorem is due to Bagemihl and Seidel [3]:

Theorem 18 (Bagemihl-Seidel, 1954). *Let ϕ be a continuous function on D , and let E be a first category set in T . Then there is a holomorphic function f in D such that, for all $\omega \in E$, we have:*

$$\lim_{r \rightarrow 1} \{f(r\omega) - \phi(r\omega)\} = 0.$$

The proof makes use of Mergelyan's Theorem.

Corollary 19. *If E is a first category subset of T , then there is a nonconstant holomorphic function with radial limit zero on E .*

Recall that there are first category sets with positive measure (even *full* measure). With this observation, the following corollaries are immediate:

Corollary 20. *There exists a set E of full measure on T and a nonconstant holomorphic function f on D with radial limit zero on E .*

Note that such a function could not be bounded (else it would violate the theorem of F. and M. Riesz).

Corollary 21. *There is a holomorphic function on D for which the set of points of T where the radial limit exists has measure zero.*

By Fatou's Theorem, the holomorphic function in the above corollary must be unbounded. In fact, the function cannot be in any of the H^p spaces.

Corollary 22. *If ϕ is a measurable function on T , then there is a holomorphic function $f : D \rightarrow \mathbb{C}$ such that $\lim_{r \rightarrow 1} f(r\omega) = \phi(\omega)$ for almost every $\omega \in T$.*

In general, the function in the above corollary cannot be bounded. However, Kahane and Katznelson [23] showed that the growth of the function at the boundary can be controlled.

Theorem 23 (Kahane-Katznelson, 1971). *If $\mu : [0, 1) \rightarrow (0, +\infty)$ is increasing and unbounded (i.e., μ is a “growth rate”), then it is possible to take f in Corollary 22 such that $|f(z)| \leq \mu(|z|)$ for every $z \in D$.*

In 1980, Rudin posed the following question (see [35, p. 414] or [36, p. 67]): Does there exist a nonconstant holomorphic function on B ($n > 1$) that has radial limit zero almost everywhere on S ? In 1987, Hakim and Sibony [20] proved the following result (which, in particular, gives an affirmative answer to Rudin’s question):

Theorem 24 (Hakim-Sibony, 1987). *There exists a set E of full measure in S with the following property: for every continuous function ϕ on B , there is a holomorphic function f in B such that*

$$\lim_{r \rightarrow 1} f(r\omega) - \phi(r\omega) = 0$$

for every $\omega \in E$.

The set E in the above theorem happens to be of first category in S (in fact, by a result that we will show, such a set *must* be of first category). Note that the theorem applies to one particular first category set E and not to *all* first category sets (as the Bagemihl-Seidel theorem does in the plane).

Corollary 25. *There exists a set E of full measure on S and a nonconstant holomorphic function f on B with radial limit zero on E .*

In 1989, Yves Dupain [14] proved a result which strengthens the above corollary. Before stating the theorem, we need the following definition. We will call a circle on S a *great circle* if it is the intersection of S with some complex line passing through the origin. Note that not every circle on S that is centered at the origin is a great circle.

Theorem 26 (Dupain, 1989). *There is a set $E \subset S$ having full (linear) measure on every great circle and a nonconstant holomorphic function f on B with radial limit zero on E .*

In fact, Dupain proved that the set E in the Hakim-Sibony theorem can be taken to be of full (linear) measure in any great circle.

3. Miscellaneous Results

In this subsection, we will state several miscellaneous results. Some of these are results about growth rates, some are about general harmonic functions (instead of holomorphic functions), some are about cluster sets, and some take the domain to be the half space or the polydisc (instead of the ball). We state them in no particular order.

The set of points of T where the radial limit of a bounded holomorphic function fails to exist must be a $\mathcal{G}_{\delta\sigma}$ set of measure zero. In 1995, Kolesnikov [24] showed that the converse is also true.

Theorem 27 (Kolesnikov, 1995). *If $E \subset T$ is a $\mathcal{G}_{\delta\sigma}$ set of measure zero, then there is a bounded holomorphic function f on D such that $\lim_{r \rightarrow 1} f(r\omega)$ exists when $\omega \in E^c$ and fails to exist when $\omega \in E$.*

The following Fatou-type theorem of Nagel and Rudin [32] concerns normal limits

of functions on a rectangle in \mathbb{C} . The important thing to note about this theorem is that the functions are not necessarily holomorphic.

Theorem 28. *For real numbers a, b, c (with $a < b$ and $c > 0$), let $A = (a, b) \times (0, c) \subset \mathbb{C}$. If $f : A \rightarrow \mathbb{C}$ is a bounded C^1 function satisfying $\partial f / d\bar{z} \in L^p(A)$ for some $p > 1$, then $\lim_{y \rightarrow 0^+} f(x + iy)$ exists for almost every $x \in (a, b)$.*

In the same paper, Nagel and Rudin proved the following theorem.

Theorem 29. *If $\phi : [a, b] \rightarrow S$ is a class C^1 curve that is nowhere complex-tangential (i.e., $\langle \phi'(t), \phi(t) \rangle \neq 0$ for all $t \in [a, b]$) and if $f : B \rightarrow \mathbb{C}$ is bounded and holomorphic, then the restricted K -limit of f exists at $\phi(t)$ for almost every $t \in [a, b]$.*

The following proposition gives a converse of the previous theorem. See [35, p. 237] for a proof.

Proposition 30. *If γ is a complex-tangential curve in S , then there is a bounded holomorphic function $f : B \rightarrow \mathbb{C}$ for which the limit along any curve in B that ends at a point of γ does not exist.*

The following theorem concerning harmonic, superharmonic, and holomorphic functions on the upper half-plane was proved in several stages (see, for example, [1], [30], and [7]) and is stated in [18].

Theorem 31. *Let M denote the upper half-plane $\{(x, t) : x \in \mathbb{R}, t > 0\}$. The following conditions are equivalent for a subset $E \subset \mathbb{R}$:*

1. *there is a harmonic function u on M such that for $x \in E$,*

$$\lim_{t \rightarrow 0^+} u(x, t) = +\infty;$$

2. there is a nonconstant holomorphic function f on M such that for $x \in E$,

$$\lim_{t \rightarrow 0^+} f(x + it) = 0;$$

3. there is a superharmonic function u on M such that for $x \in E$,

$$\lim_{t \rightarrow 0^+} u(x, t) = +\infty;$$

4. for each interval A , either $E \cap A$ is first category or there is some open subinterval $A' \subset A$ such that $E \cap A'$ has measure 0.

The following theorem concerns normal limits of *harmonic* function on the half space in \mathbf{R}^{n+1} and is due to S. Gardiner and W. Hansen [18]:

Theorem 32 (Gardiner-Hansen, 2002). *Let $n \geq 1$, and let M denote the half space $\{(x, t) : x \in \mathbf{R}^n, t > 0\}$. The following are equivalent for a set $E \subset \mathbf{R}^n$:*

1. there is a harmonic function u on M such that for $x \in E$

$$\lim_{t \rightarrow 0^+} u(x, t) = +\infty;$$

2. there is a harmonic function u on M such that for $x \in E$

$$\lim_{t \rightarrow 0^+} |u(x, t)| = +\infty$$

3. there is a continuous superharmonic function $u : M \rightarrow (-\infty, +\infty]$ such that for $x \in E$

$$\lim_{t \rightarrow 0^+} u(x, t) = +\infty;$$

4. there is an increasing sequence $\{E_k\}$ of sets such that $E = \bigcup_k E_k$ and $\lambda(E \cap V_k) = 0$ for each k , where V_k denotes the fine interior of $\overline{E_k}$.

K. Tse [38] proved the following theorem concerning growth rates to the boundary

of holomorphic functions on D .

Theorem 33 (Tse, 1970). *Let $\mu(r)$ be any positive monotone decreasing function on $[0, 1)$ such that $\lim_{r \rightarrow 1} \mu(r) = 0$. Let E be a second category subset of the unit circle T . If f is a meromorphic function in D with the property that for each $e^{i\theta} \in E$:*

$$|f(re^{i\theta})| = o(\mu(r)),$$

then $f \equiv 0$.

Barth and Schneider [4] had earlier proved the above result with the additional hypothesis that f is bounded. Tse also showed that the above theorem is sharp:

Theorem 34. *Let $\mu(r)$ be any positive monotone decreasing function on $[0, 1)$ such that $\lim_{r \rightarrow 1} \mu(r) = 0$. Let E be any first category set in T . Then there is a nonconstant holomorphic function f on D such that for each $e^{i\theta} \in E$:*

$$|f(re^{i\theta})| = o(\mu(r)).$$

Gardiner [17] proved the following result, which is similar in spirit to the Barth-Schneider theorem.

Theorem 35. *Let $f : (0, 1] \rightarrow \mathbb{R}$ be such that $f(x) \rightarrow -\infty$ as $x \rightarrow 0+$, and let u be a subharmonic function on the upper half-space M . Then the set*

$$E = \{X' \in \mathbb{R}^{n-1} : \limsup_{x \rightarrow 0+} \{u(X', x) - f(x)\} < +\infty\}$$

is of first fine category.

The following theorem concerning the existence of a certain kind of “universal” holomorphic function is due to F. Bayart [5]:

Theorem 36 (Bayart, 2005). *There is a holomorphic function $f : B \rightarrow \mathbb{C}$ with the following property: given any measurable function ϕ on S , there is a sequence $\{r_n\}_{n=1}^\infty$, $0 < r_n < 1$, that converges to 1, such that $\lim_{n \rightarrow \infty} f(r_n \omega) = \phi(\omega)$ for almost every $\omega \in S$.*

Collingwood's Maximality Theorem (see [12]) is an important theorem in the theory of cluster sets. We will state the result, but we first need some definitions. Let f be a real or complex-valued function on D . The *cluster set* of f at the point $e^{i\theta}$ (denoted by $C(f, e^{i\theta})$) is the set of all values which are limits (in the extended sense) of sequences $\{f(z_k)\}$ where $z_k \rightarrow e^{i\theta}$. Let γ_0 be a path in D with terminal point at $z = 1$. If we rotate γ_0 about the origin by an angle θ we obtain a new path γ_θ with terminal point at $z = e^{i\theta}$. We define the *partial cluster set* of f on γ_θ (denoted by $C_{\gamma_\theta}(f, e^{i\theta})$) to be the set of all values which are limits (in the extended sense) of sequences $\{f(z_k)\}$ where the z_k 's lie along γ_θ and converge to $e^{i\theta}$. A path γ_θ is called *monotonic* if it intersects every circle $|z| = r$ (with $r < 1$) in at most one point.

Theorem 37 (Collingwood's Maximality Theorem). *Suppose f is a continuous real or complex-valued function on D . Let γ_0 be a monotonic path lying in D except for its terminal point at $z = 1$, and let $\{\gamma_\theta\}$ be the family of rotations of γ_0 about the origin. Then*

$$\{\theta \in [0, 2\pi] : C_{\gamma_\theta}(f, e^{i\theta}) = C(f, e^{i\theta})\}$$

is residual in $[0, 2\pi]$.

Corollary 38. *Suppose f is a continuous real or complex-valued function on D . Let A be the set of points of T where the radial limit of f exists, and let G be the set of points of T where the general limit of f exists. Then $A \setminus G$ is of first category in T .*

Along this circle of ideas, T.J. Kaczynski [22] proved the following theorem in 1969 concerning harmonic Hardy space functions.

Theorem 39. *Let f be a harmonic (or holomorphic) function such that, for some $p > 1$*

$$\left\{ \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta : r \in [0, 1) \right\}$$

is a bounded set of numbers. Let A be the set of points of T at which f has an asymptotic value, and let G be the set of points of T at which the general limit of f exists. Then $A \setminus G$ is first category.

A proof of the following decomposition theorem can be found in [35, p. 83]

Theorem 40. *If $f : B \rightarrow \mathbb{C}$ is holomorphic, then S is the disjoint union of three sets E_K , $E_{\mathbb{C}}$, and E_N , where*

1. *f has finite K -limit for each $z \in E_K$,*
2. *$\overline{f(D_t(\omega))} = \mathbb{C}$ for every $t > 1$ and every $\omega \in E_{\mathbb{C}}$, and*
3. *E_N has measure zero.*

The following theorem was proved by M. Tsuji [39]. In Chapter V, we will prove a strengthening of part 2 of this theorem.

Theorem 41. *Let f be a bounded holomorphic function on the unit polydisc D^n . The following statements hold:*

1. *For almost every point $(e^{i\theta_1}, \dots, e^{i\theta_n}) \in T^n$, the limit of $f(z_1, \dots, z_n)$ exists as each z_k approaches $e^{i\theta_k}$ non-tangentially to $|z_k| = 1$.*
2. *If the boundary value (from part 1) is zero on a set of positive measure on T^n , then $f \equiv 0$.*

C. Baire Category

Let (X, d) be a complete metric space. We will often make no mention of the metric d and will simply use X to denote (X, d) . A set $E \subset X$ is said to be *nowhere dense* in

X if its closure has empty interior (that is, if $(\overline{E})^o = \emptyset$). Equivalently, E is *nowhere dense* in X if it is dense in no open subset of X . A set $F \subset X$ is said to be *first category* in X if $F = \cup_{j=1}^{\infty} E_j$ where each E_j is nowhere dense in X . A subset of X is said to be *second category* in X if it is not first category. A subset of X is said to be *residual* if its complement is first category.

Theorem 42 (Baire Category Theorem). *A complete metric space is second category in itself.*

Proof. Let (X, d) be a complete metric space, and let $E = \cup_{j=1}^{\infty} E_j$ where each E_j is a nowhere dense subset of X . Let B_0 be a nonempty open ball in X . We will show that $B_0 \cap (X \setminus E) \neq \emptyset$, thus showing that $E \neq X$. We may inductively choose a nested sequence of open balls $B_j := B(x_j, r_j)$ with $r_j < 1/j$ with the property that

$$\overline{B_{j+1}} \subset B_j \setminus \overline{E_{j+1}}.$$

Such a choice is possible because each $\overline{E_j}$ is closed and nowhere dense. Then $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence because, if $j, k \geq N$ then

$$d(x_j, x_k) \leq d(x_j, x_N) + d(x_N, x_k) < 2/N.$$

By the completeness of X , there is a point $x \in X$ such that $x_j \rightarrow x$. This point x is in $B_0 \cap (X \setminus E)$. Thus $E \neq X$. So X cannot be first category in itself. \square

It is clear from the definition that any subset of a first category set is again first category. So the first category sets are the “small” sets and the second category sets are the “big” sets. Residual sets are then “very big” second category sets. So the category of a set gives a notion of the size of the set. However, the following theorem shows that on the real line, category and (Lebesgue) measure are very different notions of size.

Theorem 43. *There are sets $A, B \subset [0, 1]$ such that A is first category in $[0, 1]$, B has measure zero, and $A \cup B = [0, 1]$.*

The set A can be constructed by taking a countable union of “fat Cantor sets” whose measures approach 1. Of course, similar constructions can be done on all of \mathbb{R} , on $[0, 1]^n$, on the unit sphere, etc.

CHAPTER III

THE LUSIN-PRIVALOV THEOREM

In this chapter, we will state and prove a classical uniqueness theorem of Lusin and Privalov [30] to the effect that radial limit zero sets of nonconstant holomorphic functions on the D must be locally “small” in a sense that involves both Lebesgue measure and Baire category. The converse of this result was proved by J.E. McMillan [31] and R. Berman [7]; Berman’s method of proof will be presented here. We will prove a higher dimensional version of the Lusin-Privalov result, but we will show that the converse does not hold (when $n \geq 2$).

A. Preliminaries

The statement of the Lusin-Privalov Theorem makes use of the concept of “metric density.” A set $E \subset T$ is said to be *metrically dense* in a connected open set $A \subset T$ if $E \cap G$ has positive outer measure for every non-empty open subset G of A . We use the same terminology if E and A are subsets of S instead of T .

In order to prove the Lusin-Privalov Theorem, we will need Carathéodory’s Theorem, Egorov’s Theorem, and Lindelöf’s Theorem. We will prove Lindelöf’s here but will only give references for the proofs of the other results.

A *Jordan curve* is a continuous function $\gamma : [0, 1] \rightarrow \mathbb{C}$ such that $\gamma(0) = \gamma(1)$ and with the property that $\gamma|_{[0,1)}$ is one-to-one. By a slight abuse of terminology, we will also refer to the set $\{\gamma(t) : t \in [0, 1]\}$ as a Jordan curve.

Theorem 44 (Carathéodory). *Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be bounded domains that are each bounded by finitely many Jordan curves, and let ϕ be a biholomorphic mapping of Ω_1 onto Ω_2 . Then there is a continuous bijection $\hat{\phi} : \overline{\Omega}_1 \rightarrow \overline{\Omega}_2$ with the property that*

$$\hat{\phi}|_{\Omega_1} = \phi.$$

For a proof of Carathéodory's Theorem, see [27, pp. 110-118].

Let (X, \mathcal{A}, μ) be a measure space (i.e., X is a set, \mathcal{A} is a σ -algebra on X , and μ is a measure on \mathcal{A}), and let $\{f_j\}_{j=1}^{\infty}$ be a sequence of measurable functions on X that are finite almost everywhere. Then $\{f_j\}_{j=1}^{\infty}$ is said to converge *almost uniformly* on X to a function f if for every $\epsilon > 0$ there is a measurable set E such that $\mu(X \setminus E) < \epsilon$ and such that $\{f_j\}_{j=1}^{\infty}$ converges uniformly to f on E .

Theorem 45 (Egorov). *Let (X, \mathcal{A}, μ) be a measure space such that $\mu(X) < \infty$, and let $\{f_j\}_{j=1}^{\infty}$ be a sequence of measurable functions that are finite almost everywhere. If $\{f_j\}_{j=1}^{\infty}$ converges almost everywhere to f on X , then $\{f_j\}_{j=1}^{\infty}$ converges almost uniformly to f on X .*

For a proof of Egorov's Theorem, see [16, p. 62].

Theorem 46 (Lindelöf, 1915). *Suppose f is a bounded holomorphic function on D and $\gamma : [0, 1) \rightarrow D$ is a continuous curve such that $\gamma(t) \rightarrow \omega \in T$ as $t \rightarrow 1$. If*

$$\lim_{t \rightarrow 1} f(\gamma(t)) = L$$

exists, then f has non-tangential limit L at ω .

We follow the method of proof found in [35, p. 168]

Proof. We assume, without loss of generality, that $\omega = \mathbf{1} := (1, 0, \dots, 0)$, that $|f|$ is bounded by 1 on D , and that $L = 0$. Let

$$\Omega = \{z \in \mathbb{C} : |\operatorname{Re}(z)| < 1\},$$

and, for $z \in D$, set

$$\phi(z) = \frac{i}{\pi} \log \left(\frac{1+z}{1-z} \right)^2$$

Then ϕ is a biholomorphic mapping of D onto Ω with the property that $\phi(0) = 0$. Also, if we set $\Gamma = \phi \circ \gamma$, then $\lim_{t \rightarrow 1} \operatorname{Im} \Gamma(t) \rightarrow +\infty$. Set $F = f \circ \phi^{-1}$. Then F is a bounded holomorphic function on Ω (with bound 1) such that $\lim_{t \rightarrow 1} F(\Gamma(t)) = 0$. Fix $\delta \in (0, 1)$. It suffices to show that $F(x + iy)$ converges uniformly to zero as $y \rightarrow +\infty$ for $|x| \leq 1 - \delta$.

Let $\epsilon \in (0, 1)$. Since $\lim_{t \rightarrow 1} F(\Gamma(t)) = 0$, we may choose a real number $y > \operatorname{Im} \Gamma(0)$ large enough that $|F(\Gamma(t))| < \epsilon$ if $\operatorname{Im} \Gamma(t) \geq y$. We will show that if $|x| \leq 1 - \delta$, then $|F(x + iy)| \leq \epsilon^{\delta/4}$. We may assume, without loss of generality, that $y = 0$.

Let t_0 be such that $\operatorname{Im} \Gamma(t_0) = 0$ but such that $\operatorname{Im} \Gamma(t) > 0$ for $t_0 < t < 1$. Define

$$E = \{\Gamma(t) : t_0 \leq t < 1\},$$

and

$$\overline{E} = \{\overline{\Gamma(t)} : t_0 \leq t < 1\}.$$

Let x_0 be the unique point where $E \cup \overline{E}$ intersects the real axis.

Suppose that $x \in (x_0, 1 - \delta]$. (The case $x \in [-(1 - \delta), x_0]$ is similar and will be dealt with later.) For $r \in (0, +\infty)$ and $z \in \Omega$, define

$$G_r(z) = \frac{F(z)\overline{F(\bar{z})}\epsilon^{(1+z)/2}}{1 + r(1 + z)}.$$

G_r is bounded on Ω since $|F| \leq 1$ on Ω , $|\epsilon^{(1+z)/2}| \leq 1$ on Ω , and $|1 + r(1 + z)| \geq 1$ on Ω . Also, G_r is holomorphic. Note that $|F(z)| < \epsilon$ for $z \in E$, and $|\overline{F(\bar{z})}| < \epsilon$ for $z \in \overline{E}$. So if $z \in E \cup \overline{E}$, then $|G_r(z)| < \epsilon$. Because of the factor $\epsilon^{(1+z)/2}$, the boundary values of $|G_r|$ on the right edge of Ω are also less than ϵ . Because of the denominator, we have that if $|\operatorname{Im} z|$ is sufficiently large, then $|G_r(z)| < \epsilon$. By the maximum modulus principle, we have that $|G_r(x)| < \epsilon$. If we let $r \rightarrow 0$, we get

$$|F(x)|^2 \epsilon^{(1+x)/2} \leq \epsilon.$$

Thus

$$|F(x)|^2 \leq \epsilon^{(1-x)/2} \leq \epsilon^{\delta/2},$$

since $\delta \leq 1 - x$ and $0 < \epsilon < 1$. Thus we have the uniform bound $|F(x)| \leq \epsilon^{\delta/4}$ for $x \in (x_0, 1 - \delta]$. By similar methods, we may show the same bound for $x \in [-(1 - \delta), x_0]$. \square

B. The Lusin-Privalov Theorem in One Dimension

The following classical theorem is due to Lusin and Privalov [30, p. 187]. We present the proof from that paper.

Theorem 47. *Suppose a set $E \subset T$ is both metrically dense and second category in some open arc $A \subset T$. If $f : D \rightarrow \mathbb{C}$ is holomorphic and $f^*(\omega) = 0$ for each $\omega \in E$, then $f \equiv 0$.*

Proof. Let $E \subset T$ be both metrically dense and second category in an open arc $A \subset T$, and let $f : D \rightarrow \mathbb{C}$ be holomorphic with $f^*(\omega) = 0$ for each $\omega \in E$. For each $j \in \mathbb{N}$, let

$$f_j(z) = f\left(\frac{j-1}{j}z\right).$$

Each f_j is defined (and holomorphic) in some neighborhood of \overline{D} . Also, for each $z \in E$, $f_j(z) \rightarrow 0$ as $j \rightarrow \infty$.

Fix $\epsilon > 0$. For $z \in E$, let $j(z)$ be the smallest natural number such that $|f_j(z)| < \epsilon$ for all $j > j(z)$. Let E_k be the set of points z for which $j(z) \leq k$. Then $E_k \subset E_{k+1}$ for each $k \in \mathbb{N}$, and

$$E = \bigcup_{k=1}^{\infty} E_k.$$

Since E is second category in T , $\exists i \in \mathbb{N}$ such that E_i is second category in T .

So there is an open arc $A' \subset A$ on which E_i is dense. Thus, the sets E_{i+1}, E_{i+2}, \dots

are also dense in A' . Since $\lambda(A' \cap E) > 0$, there is a $k \geq i$ such that $\lambda(A' \cap E_k) > 0$.

Let $M_k = A' \cap E_k$. Thus M_k is dense in A' and $\lambda(M_k) > 0$.

If $z \in M_k$, $|f_j(z)| < \epsilon$ for $j > k$. Let ϵ_1 be a positive number greater than

$$|f_1(z)|, |f_2(z)|, \dots, |f_k(z)|$$

for every $z \in M_k$. Let $c = \max\{\epsilon_1, \epsilon\}$. Then

$$|f_j(z)| < c$$

for every $j \in \mathbb{N}$ and every $z \in M_k$. Since each f_j is continuous on A' , and since M_k is dense on A' , then

$$|f_j(x)| < c$$

for every $j \in \mathbb{N}$ and every $x \in A'$. By using the change of variables $z = \frac{j-1}{j}x$, we transform the arc A' into an arc A'_j with radius $1 - 1/j$. After this transformation, we have

$$f(z) = f_j(x).$$

It follows that $|f(z)| < c$ for all $z \in A'_j$ and all $j \in \mathbb{N}$.

Let a and b be the endpoints of the arc A' . We may always assume, without loss of generality, that $a, b \in M_k$. The function f is bounded on the rays $0a$ and $0b$, since it tends to zero along these rays. Let a'_j and b'_j be the endpoints of the arc A'_j . Since, for every $j \in \mathbb{N}$, f is bounded by c on the contour $0a'_jb'_j0$, then f is also bounded by c on the open sector $G := 0ab0$.

By the Riemann Mapping Theorem, there is a biholomorphic mapping ϕ of D onto G . By Carathéodory's Theorem, there is a continuous one-to-one function $\hat{\phi} : \overline{D} \rightarrow \overline{G}$ such that $\hat{\phi}|_D = \phi$.

Since the set M_k has positive measure in T , then

$$M := \hat{\phi}^{-1}(M_k)$$

also has positive measure in T .

By Lindelöf's Theorem, since $g := f \circ \phi$ is bounded, g has non-tangential limit zero at every point of M . We will show that $g \equiv 0$. From a point $z \in M$ draw (in the interior of D) two rays, each having an angle of 45° with the tangent to z . For $n \in \mathbb{N}$, let $S_{n,z}$ be the intersection of $D(z, 1/n)$ with the “cone” determined by these two rays. So $S_{n,z}$ is a sector of $D(z, 1/n)$. Define $f_n : M \rightarrow \mathbb{R}$ by

$$f_n(z) = \max\{|f(w)| : w \in S_{n,z}\}.$$

Each f_n is finite and measurable, and $f_n(z) \rightarrow 0$ as $n \rightarrow \infty$ for each $z \in M$. So, by Egorov's Theorem, since $\lambda(M) > 0$, there is a perfect set (i.e., a closed set with no isolated points) P with $\lambda(P) > 0$ on which the convergence of f_n to zero is uniform.

We now construct a curve in \overline{D} in the following way. The curve will contain all of P , and for each component of the open set $T \setminus P$ it will contain a pair of line segments from the endpoints (of that component) having angles of 45° with the tangents to T at the respective endpoints. In this way, we obtain a closed, rectifiable curve. Let K be the domain whose boundary is this curve.

By the Riemann Mapping Theorem, there is a biholomorphic mapping ψ of D onto K . Let $\hat{\psi}$ be the continuous and one-to-one extension of ψ to \overline{D} that is guaranteed by Caratheodory's Theorem. Let

$$P_1 = \hat{\psi}^{-1}(P).$$

Then P_1 is perfect and has positive measure.

The function $g \circ \psi$ is holomorphic on D , and it has a continuous extension to \overline{D} ,

so we consider its domain to be \overline{D} . Moreover, $g \circ \psi$ is equal to zero on P_1 . Since, it is continuous on a compact set, $g \circ \phi$ attains a maximum value. We assume, without loss of generality, that this value is less than 1.

Let $u = \log |g \circ \phi|$. Then u is a negative subharmonic function on D that takes the value $-\infty$ on the set $P_1 \subset T$. Let $\rho \in (0, 1)$. We use the Poisson integral to estimate the value of u at a point $r_0 e^{i\theta_0} \in D(0, \rho)$:

$$u(r_0 e^{i\theta_0}) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\rho e^{i\theta}) \frac{\rho^2 - r_0^2}{\rho^2 + r_0^2 - 2\rho r_0 \cos(\theta - \theta_0)} d\theta.$$

The factor $\frac{\rho^2 - r_0^2}{\rho^2 + r_0^2 - 2\rho r_0 \cos(\theta - \theta_0)}$ is positive and is larger than a fixed constant greater than zero. Moreover, $u(\rho e^{i\theta})$ is negative and tends uniformly to $-\infty$ on the set P_1 as $\rho \rightarrow 1$. So $u(r_0 e^{i\theta_0})$ tend to $-\infty$. Thus $g(r_0 e^{i\theta_0}) = 0$. But since $r_0 e^{i\theta_0}$ was arbitrary in $D(0, \rho)$, we conclude that $g|_{D(0, \rho)} \equiv 0$. Thus, by the identity theorem, $g \equiv 0$. So $f|_G \equiv 0$. Using the identity theorem again, we thus obtain $f \equiv 0$.

□

C. The McMillan-Berman Converse

Lusin and Privalov gave partial results in the direction of a converse of Theorem 47, but they were unable to establish the full converse. J.E. McMillan [31] and R. Berman [7] showed that the following full converse of the Lusin-Privalov theorem holds.

Theorem 48 (McMillan, 1966; Berman, 1983). *If for any open arc $A \subset T$, a set $E \subset T$ is not both metrically dense and second category A , then there exists a nonconstant holomorphic function f on D such that $f^*(\omega) = 0$, $\omega \in E$.*

In order to present Berman's method of proof, we will need a few lemmas. The first lemma is due to Privalov [33].

Lemma 49 (Privalov, 1956). *Let $E \subset T$ be a set of measure zero. Then there exists a nonconstant bounded holomorphic function on D with radial limit zero on E .*

Proof. Since $E \subset T$ has measure zero, for each $t \in \mathbb{N}$ there is an open set $E_t \subset T$ that contains E and has measure less than $\frac{1}{2^t}$. Let

$$\chi_{E_t}(\omega) = \begin{cases} 1 & \text{if } \omega \in E_t \\ 0 & \text{if } \omega \notin E_t \end{cases}$$

and define $g : T \rightarrow \mathbb{R}^+$ by

$$g(\omega) = \sum_{t=1}^{\infty} \chi_{E_t}(\omega).$$

Then $g \in L_1(T)$ with $\|g\|_{L_1} \leq 1$. Define $u : D \rightarrow \mathbb{R}^+$ by

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} d\phi.$$

Then u is a positive (finite) harmonic function on D with the property that

$$\lim_{r \rightarrow 1^-} u(r\omega) = +\infty$$

for each $\omega \in E$. Since D is simply connected, there is a function $v : D \rightarrow \mathbb{R}$ such that $u + iv$ is holomorphic. Let

$$f = e^{-u-iv}.$$

Then f is a holomorphic function on D . But $|f(z)| = e^{-u(z)}$, so

$$\lim_{r \rightarrow 1^-} f(r\omega) = 0$$

for each $\omega \in E$. Clearly, f is nonconstant since it does not take the value of zero inside the disc but it has radial limit zero on E (unless, of course, $E = \emptyset$, in which case the result is trivial). Finally, $|f|$ is bounded (by 1) since $u \geq 0$. \square

The following decomposition lemma is due to Berman [7].

Lemma 50. *Suppose a set $E \subset T$ has the property that for every nonempty open arc $A \subset T$, E is not both metrically dense and second category in A . Then there is a closed set $F \subset T$ such that $E \cap F$ is of first category and $E \setminus F$ has measure zero.*

Proof. Let F be the set of $\omega \in T$ such that the set $E \cap A$ has positive outer measure for every open arc A containing ω .

Let $\{x_n\}_{n=1}^\infty$ be a sequence of points in F that converges to some point $x \in T$. Any open arc containing x must contain some x_j . So E has positive measure in this arc. Thus $x \in F$. This shows that F is closed.

Since $F \setminus F^\circ$ is nowhere dense in T , to show that $E \cap F$ is first category, it suffices to show that $E \cap F^\circ$ is first category. Let A be an open component of F° (i.e., A is an open arc). By the definition of F , E is metrically dense in A . So $E \cap A$ must be first category. But since F° has only countably many components, $E \cap F^\circ$ is first category.

We now show that $E \setminus F$ has measure zero. By the definition of F , for each $\omega \in T \setminus F$, there is an open arc A_ω containing ω such that $E \cap A_\omega$ has measure zero. Then $\{A_\omega\}_{\omega \in T \setminus F}$ is an open cover of $T \setminus F$. Let $\{A_{\omega_k}\}_{k=1}^\infty$ be a *countable* subcover. But since $E \cap A_{\omega_k}$ has measure zero for each $k \in \mathbb{N}$, then $E \setminus F \subset \bigcup_{k=1}^\infty (E \cap A_{\omega_k})$ has measure zero as well.

□

The following lemma is a small improvement of Corollary 19. This lemma is also due to Berman [7], and we will follow his method of proof.

Lemma 51. *Let E be a first category subset of T . Then there is a nonconstant holomorphic function $f : D \rightarrow \mathbb{C}$ which is continuous up to $\overline{D} \setminus \overline{E}$ and such that $f^*(\omega) = 0$ for each $\omega \in E$.*

Proof. We may, of course, assume that E is nonempty. Let

$$R = D \cup \{D(0, 2) \setminus \{0\} : z/|z| \in T \setminus \overline{E}\}.$$

Since E is first category, we may write $E = \bigcup_{j=1}^{\infty} E_j$ where each E_j is nowhere dense in T . Let $F_j = \bigcup_{k=1}^j \overline{E}_k$. For each $j \in \mathbb{N}$, F_j is a closed nowhere dense subset of T , and $F_j \subset F_{j+1}$. We also have $E \subset \bigcup_{j=1}^{\infty} F_j \subset \overline{E}$. For $j \in \mathbb{N}$, let

$$W_j = \{\omega \in T : \text{dist}(\omega, \overline{E}) \geq 1/j\}.$$

Then each W_j is a compact subset of $T \setminus \overline{E}$, and $\bigcup_{j=1}^{\infty} (W_j)^{\circ} = T \setminus \overline{E}$. We also clearly have $W_j \subset W_{j+1}$. Let

$$S_j = \overline{D(0, 1 - 1/j)} \cup \{D(0, 2) \setminus \{0\} : z/|z| \in W_j\}$$

and

$$T_j = \{z \in \overline{D} \setminus D(0, 1 - 1/j) : z/|z| \in F_j\}.$$

For $j \in \mathbb{N}$, let $K_j = S_j \cup T_j$, and note that each K_j (being the union of two compact sets) is compact. For $j \in \mathbb{N}$, define a non-negative function $h_j : K_j \rightarrow \mathbb{R}$ by

$$h_j(z) = \begin{cases} 0 & \text{if } z \in S_j, \\ j2^j[|z| - (1 - 1/j)] & \text{if } z \in T_j. \end{cases}$$

Note that each h_j is continuous on K and holomorphic on K° . Since $\mathbb{C} \setminus K_j$ is connected, we may use Mergelyan's Theorem to obtain a holomorphic polynomial p_j with the property that for each $z \in K_j$,

$$|p_j(z) - h_j(z)| < 1/2^j.$$

Note that $\sum_{j=1}^{\infty} p_j$ converges uniformly on compact subsets of R to a holomorphic

function g . Let $f : D \rightarrow \mathbb{C}$ be defined by $f(z) = e^{g(z)}$. It is straightforward to check that f satisfies the required properties.

□

We are now in a position to give Berman's proof of Theorem 48.

Proof. Let F be the closed subset of T that is guaranteed by Lemma 50. Then by Lemma 49, since $E \setminus F$ has measure zero, there is a nonconstant bounded holomorphic function $g : D \rightarrow \mathbb{C}$ such that $g^*(\omega) = 0$ for each $\omega \in E \setminus F$. Also, since $E \cap F$ is of first category, there is a nonconstant holomorphic function $h : D \rightarrow \mathbb{C}$ such that $h^*(\omega) = 0$ for each $\omega \in E \cap F$ and such that h is analytic at each point of $T \setminus (\overline{E \cap F}) \supset T \setminus \overline{F} = T \setminus F$. Finally, let $f = gh$.

□

D. The Luzin-Privalov Theorem in Higher Dimensions

In this section, we prove a higher dimensional version of the Luzin-Privalov Theorem. But we first prove the following lemma:

Lemma 52. *Let f be a holomorphic function on B . Then $E = \{\omega \in S : f^*(\omega) = 0\}$ is an $\mathcal{F}_{\sigma\delta}$ subset of S . (In particular, E is measurable.)*

It is important to note that the set E is not simply a radial limit zero set of f , but it is the *precise* radial limit zero set of f . That is, it is not just some particular set where f happens to have radial limit zero, but it is the set of *all* points where f has radial limit zero.

Proof. For $j, k \in \mathbb{N}$, let $F_{j,k} = \{\omega \in S : |f(r\omega)| \leq 1/j \text{ when } k/(k+1) < r < 1\}$. Since f is continuous, each $F_{j,k}$ is closed. Writing $E = \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} F_{j,k}$, we obtain the desired conclusion.

□

The proof that Lusin and Privalov gave of Theorem 47 does not immediately generalize to higher dimensions because it uses the Riemann mapping theorem. However, making a few modifications, we give a proof in a similar spirit.

Theorem 53. *Suppose f is a nonconstant holomorphic function on B such that $f^*(\omega) = 0$ for each ω in some set $E \subset S$. Then given a non-empty open set $A \subset S$, E is not both metrically dense and second category in A .*

Proof. Let f be a nonconstant holomorphic function on B with $f^*(\omega) = 0$ for each ω in some set $E \subset S$. (Since there is a *measurable* set containing E at which f has radial limit zero, we may assume without loss of generality that E is measurable.) Suppose, by way of contradiction, that there is an open set $A \subset S$ such that E is both metrically dense and second category in A . For $k \in \mathbb{N}$, let

$$F_k = \{\omega \in \bar{A} : |f(r\omega)| \leq k, \forall r \in [0, 1)\}.$$

Note that each F_k is closed in S and that $\bigcup_{k=1}^{\infty} F_k \supset E \cap A$. By our assumption, $E \cap A$ is a second category subset of \bar{A} . So by the Baire Category Theorem, $\exists j \in \mathbb{N}$ such that F_j has non-empty interior. Thus, setting

$$G = (F_j)^\circ \cap A,$$

we have $|f(r\omega)| \leq j$ for $\omega \in G$ and $r \in [0, 1)$. By intersecting with a ball, we may assume without loss of generality that G is the intersection of S with some small open ball centered at a point of S . Since G is an open subset of A , then by the metric density of E in A , $\sigma(E \cap G) > 0$, where σ is Lebesgue measure on S .

For $t \in \mathbb{N}$, set

$$G_t = \{r\omega : \omega \in G, 0 < r < \frac{t}{t+1}\}$$

and

$$E_t = \left\{ \frac{t}{t+1} \omega : \omega \in G \cap E \right\}.$$

Also, set $G_\infty = \{r\omega : \omega \in G, 0 < r < 1\}$ and $E_\infty = G \cap E$. Note that $E_t \subset \partial G_t$. Let P_t be the Poisson kernel for G_t . Fix $z_0 \in G_1$ and set $M = \inf\{P_t(z_0, w) : t \in \mathbb{N} \cup \{\infty\}, w \in \partial G_t\}$. Since $P_t(\cdot, w)$ is positive and continuous, we have by scaling that $M > 0$.

Since f is holomorphic, $\log |f|$ is subharmonic. Also, since $|f|$ is bounded (on G_∞), we assume without loss of generality that it is bounded by 1 (hence $\log |f| \leq 0$ on G_∞).

For $t \in \mathbb{N}$, define $f_t : G \rightarrow \mathbb{C}$ by

$$f_t(z) = f\left(\frac{t}{t+1}z\right).$$

Then $\{f_t(z)\}_{t=1}^\infty$ converges to 0 on $E \cap G$. By Egorov's Theorem, since $\sigma(E \cap G) > 0$, there is a subset $E' \subset E \cap G$ with $\sigma(E') > 0$ such that $\{f_t\}$ converges *uniformly* to 0 on E' .

Let σ_t be Lebesgue measure on ∂G_t . Let $E'_t = \frac{t}{t+1}E'$. Note $E'_t \subset E_t$. Putting everything together, we have

$$\begin{aligned}
\log |f(z_0)| &\leq \int_{\partial G_t} P_t(z_0, w) \log |f(w)| d\sigma_t(w) \\
&\leq \int_{E_t} P_t(z_0, w) \log |f(w)| d\sigma_t(w) \\
&\leq \int_{E'_t} P_t(z_0, w) \log |f(w)| d\sigma_t(w) \\
&\leq \int_{E'_t} M \log |f(w)| d\sigma_t(w) \\
&\leq \int_{E'} M \log \left| f \left(\frac{t}{t+1} w \right) \right| d\sigma(w) \\
&= \int_{E'} M \log |f_t(w)| d\sigma(w) \\
&\rightarrow -\infty \quad (t \rightarrow 1)
\end{aligned}$$

The limiting argument in the last step follows from the uniform convergence of f_t to zero on E' . Thus $f(z_0) = 0$. Since z_0 was arbitrary in G_1 , we have $f \equiv 0$ on G_1 . The identity theorem then gives $f \equiv 0$ on B , contradicting the assumption that f is nonconstant.

□

E. Failure of the Converse in Higher Dimensions

In contrast to the $n = 1$ case, the converse is not true if $n \geq 2$. The following example exhibits a set $E \subset S$ that is *both* measure zero and first category for which there does *not* exist a nonconstant holomorphic function with radial limit zero on E .

Example:

Let $\{a_j\}_{j=1}^\infty$ be a sequence of points in S that is dense in S . For each j , let L_j be the (unique) complex line containing the origin 0 and a_j . Let $C_j = L_j \cap S$. Note that each C_j is a circle. Let $E = \bigcup_{j=1}^\infty C_j$. Then E is first category in S (being a countable

union of nowhere dense sets). Also, E is measure zero in S (being a countable union of measure zero sets). Suppose $f : B \rightarrow \mathbb{C}$ is holomorphic with radial limit zero on E . Then, by the Lusin-Privalov theorem, f must be identically zero on each $L_j \cap B$. But note that $\bigcup_{j=1}^{\infty} L_j \cap B$ is dense in B . Thus, by continuity, $f \equiv 0$ on B . Note that E is not only “locally small either in the sense of measure or the sense of category,” but it is *globally* small in *both* senses. So the converse of the Lusin-Privalov theorem fails in a rather strong way when $n \geq 2$.

CHAPTER IV

CONSTRUCTION OF A RESIDUAL RADIAL LIMIT ZERO SET

The Bagemihl-Seidel Theorem can be used to construct nonconstant holomorphic functions on D having radial limit zero almost everywhere on T . In [35], Rudin posed the following question: Does there exist a nonconstant holomorphic function on B (in \mathbb{C}^n , $n \geq 2$) having radial limit zero almost everywhere on S ? It follows from results proved by Hakim and Sibony [20] and Dupain [14] that such functions exist.

Since, in dimension one, the characterization of radial limit zero sets of nonconstant holomorphic functions on D involves both measure and category, it is natural to ask the following question in \mathbb{C}^n : Does there exist a nonconstant holomorphic function on B having radial limit zero on a *residual* subset of S ? In this chapter, we will construct such a function.

Berman [6] proved the following theorem.

Theorem 54. *Suppose E is a second category subset of S , μ is a positive function on $[0, 1)$ with the property that $\lim_{r \rightarrow 1^-} \mu(r) = 0$, and f is a holomorphic function on B such that for each $\omega \in E$,*

$$\limsup_{r \rightarrow 1} \frac{|f(r\omega)|}{\mu(r)} < +\infty.$$

Then $f \equiv 0$.

We will give Berman's method of proof.

Proof. For $\omega \in S$, define $R_\omega = \{r\omega : 0 \leq r \leq 1\}$. Let $Z_\mu(f) = \{\omega \in S : \limsup_{r \rightarrow 1} \frac{|f(r\omega)|}{\mu(r)} < +\infty\}$. For $j \in \mathbb{N}$, let

$$F_j = \{\omega \in S : |f(z)| \leq j\mu(|z|) \text{ for } z \in R_\omega \text{ and } 1 - 1/j \leq |z| < 1\}.$$

Each F_j is closed, and $\bigcup_{j \in \mathbb{N}} F_j \supset Z_\mu(f)$. So, since $Z_\mu(f)$ is second category in S , there is a $j_0 \in \mathbb{N}$ such that F_{j_0} contains a non-empty open subset A of S . Since $\bigcup_{\omega \in A} R_\omega \cap \{1 - 1/j_0 \leq |z| < 1\}$ is a (relative) neighborhood of each point of A , we have that f is continuously zero at each point of A .

Let $z_0 \in B$ and $\omega_0 \in A$. The intersection with B of the complex line through z_0 and ω_0 is a disc. If we restrict f to this disc, then we get a holomorphic function that is continuously zero on some open arc of the boundary. So by the Schwarz Reflection Principle and the identity theorem, the restriction is identically zero. In particular, $f(z_0) = 0$. But z_0 was arbitrary in B , so $f \equiv 0$.

□

A. Constructing a Residual Radial Limit Zero Set in One Dimension

Theorem 54 indicates that any nonconstant holomorphic function on B with radial limit zero on a residual subset of S must have very strange boundary behavior. On the other hand, Collingwood's Maximality Theorem (Theorem 37) guarantees that if such a function exists, then the same function must have *general* limit zero on a residual set. It may at first seem that it is impossible for such functions to exist. However, since there are residual sets having measure zero, we have by Lemma 49 that such functions do indeed exist, at least in dimension one.

However, the proof of Lemma 49 does not extend to higher dimensions. The proof breaks down when finding the “harmonic conjugate” v . In contrast to the $n = 1$ case, harmonic functions on B (in \mathbb{C}^n , $n \geq 2$) are not necessarily real parts of holomorphic functions on B . Of course, the fact that the *proof* of Lemma 49 does not extend to higher dimensions does not rule out the possibility that the result still holds, but with a different proof. However, a counter-example is given by the example

at the end of Chapter III.

1. A New Method in Dimension One

We now present a new construction of a residual radial limit zero set in dimension one. We will later show how this result may be extended to higher dimensions.

Theorem 55. *There is a residual set $E \subset T$ and a nonconstant holomorphic function $f : D \rightarrow \mathbb{C}$ such that $f^*(\omega) = 0$ for each $\omega \in E$.*

Proof. For $\theta \in \mathbb{R}$, define $g_\theta : \overline{D} \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$g_\theta(z) = \operatorname{Re} \left(\frac{1}{1 - (e^{-i\theta} z)^2} \right).$$

On D , g_θ is a positive harmonic function. For $z \in \overline{D}$, $g_\theta(z) = +\infty$ if and only if $z = \pm e^{i\theta}$.

For $k, t \in \mathbb{N}$, let

$$M_{k,t} = \max\{g_\theta(z) : z \in \overline{D(0, 1 - 1/(k+t))}\}.$$

$M_{k,t}$ is independent of θ , so it is well-defined. Choose $r_{k,t} > 0$ small enough that

$$\frac{g_\theta(z)}{M_{k,t} 2^{k+t}} > 1$$

for $z \in D(e^{i\theta}, r_{k,t}) \cup D(-e^{i\theta}, r_{k,t})$. Again, $r_{k,t}$ does not depend on θ , so it is well-defined.

Let $\{a_k\}_{k=1}^\infty$ be a dense sequence in T . Let

$$E_t = \bigcup_{k=1}^\infty T \cap D(a_k, r_{k,t})$$

and

$$E = \bigcap_{t=1}^\infty E_t.$$

Note that E_t^c is nowhere dense in T . So E is residual in T . For each k , let θ_k be the unique point in $[0, 2\pi)$ such that $a_k = e^{i\theta_k}$. We claim that

$$u(z) = \sum_{t \in \mathbb{N}} \sum_{k \in \mathbb{N}} \frac{g_{\theta_k}(z)}{M_{k,t} 2^{k+t}}$$

is a positive harmonic function on D having radial limit $+\infty$ on E .

By the way that $M_{k,t}$ was defined, the double sum converges uniformly on compact subsets of D . Since each of the terms being summed is positive and harmonic on D , u is also positive and harmonic on D . For fixed t , the inner sum is greater than 1 on E_t , a (relative) neighborhood of E . So u has radial limit $+\infty$ on E (and, in fact, on $E \cup -E$).

Let $v : D \rightarrow \mathbb{R}$ be a harmonic conjugate of u . Define $f(z) = e^{-(u(z)+iv(z))}$, and note that $|f(z)| = e^{-u(z)}$. So f is a holomorphic function on D with radial limit zero on E .

□

B. Constructing a Residual Radial Limit Zero Set in Higher Dimensions

We will now show that Theorem 55 may be extended to higher dimensions using a similar method of proof.

Theorem 56. *There is a residual subset E of the unit sphere S and a nonconstant holomorphic function $f : B \rightarrow \mathbb{C}$ with $\lim_{r \rightarrow 1^-} f(r\omega) = 0$ for each $\omega \in E$.*

Proof. For $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$, define $g_\theta : \bar{B} \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$g_\theta(z_1, \dots, z_n) = \operatorname{Re} \left(\frac{1}{1 - \sum_{j=1}^n (e^{-i\theta_j} z_j)^2} \right).$$

We show in Lemma 57 that g_θ is positive and finite on the (open) unit ball. Since g_θ is the real part of a holomorphic function, it is pluriharmonic. Denote the $\{+\infty\}$ -

set in \overline{B} of a fixed g_θ by C_θ . (It turns out that each C_θ is a unit sphere (centered at the origin) of real dimension $n - 1$. But we will not need this fact.)

For $k, t \in \mathbb{N}$, let $M_{k,t} = \max\{g_\theta(z) : z \in \overline{B(0, 1 - 1/(k+t))}\}$. $M_{k,t}$ is independent of the choice of $\theta \in \mathbb{R}^n$, so it is well defined. For $r \in (0, 1)$, define tube-like neighborhoods $A_\theta(r)$ of C_θ as follows:

$$A_\theta(r) = \{z \in \mathbb{C}^n : \text{dist}(z, C_\theta) < r\}.$$

Now choose $r_{k,t} > 0$ small enough that

$$\frac{g_\theta(z)}{M_{k,t}2^{k+t}} > 1$$

for $z \in B \cap A_\theta(r_{k,t})$. Again, $r_{k,t}$ does not depend on θ , so it is well-defined.

Having chosen $r_{k,t}$, choose $r'_{k,t} > 0$ small enough that for $\theta \in \mathbb{R}^n$,

$$\bigcup \{C_\phi : \phi \in B_{\mathbb{R}^n}(\theta, r'_{k,t})\} \subset A_\theta(r_{k,t}).$$

Once again, this choice is independent of $\theta \in \mathbb{R}^n$.

Let $\{a_k\}_{k=1}^\infty$ be a dense sequence in $[0, \pi)^n$. For $t \in \mathbb{N}$, let

$$P_t = \{\theta : \theta \in B_{\mathbb{R}^n}(a_k, r'_{k,t}) \text{ for some } k \in \mathbb{N}\} \cap [0, \pi)^n,$$

and let

$$E = \bigcap_{t \in \mathbb{N}} \bigcup_{\theta \in P_t} C_\theta.$$

We show in Lemma 58 that E is residual in S .

Let

$$u(z) = \sum_{t \in \mathbb{N}} \sum_{k \in \mathbb{N}} \frac{g_{a_k}(z)}{M_{k,t}2^{k+t}}.$$

By the way that $M_{k,t}$ was defined, the double sum converges uniformly on compact subsets of B . For fixed t , the inner sum is greater than 1 in a (relative) neighborhood

of E (i.e., it is greater than 1 on $B \cap \bigcup_{k=1}^{\infty} A_{a_k}(r_{k,t})$ which is a relative neighborhood of $\bigcup_{\theta \in P_t} C_{\theta} \supset E$). The intersection of finitely many relative neighborhoods of E is again a relative neighborhood of E . So for any $j \in \mathbb{N}$, there is a relative neighborhood of E on which $u \geq j$. Thus u is a positive pluriharmonic function on B with $\lim_{r \rightarrow 1^-} u(r\omega) = +\infty$, $\omega \in E$. In fact, much more than this is true: If $\omega \in E$, then $\lim_{z \rightarrow \omega} u(z) = +\infty$. Let v be a pluriharmonic conjugate of u . Finally, define $f(z) = e^{-(u(z)+iv(z))}$, and note that $|f(z)| = e^{-u(z)}$. So f is a holomorphic function on B with $\lim_{r \rightarrow 1^-} f(rz) = 0$, $z \in E$. Again, much more than this is true: If $\omega \in E$, then $\lim_{z \rightarrow \omega} f(z) = 0$.

□

Lemma 57. *The function g_{θ} is positive and finite on B for each fixed $\theta \in \mathbb{R}^n$.*

Proof. Let $z \in B$. Then

$$\left| \sum_{j=1}^n (e^{-i\theta_j} z_j)^2 \right| \leq \sum_{j=1}^n |e^{-i\theta_j} z_j|^2 = \sum_{j=1}^n |z_j|^2 < 1.$$

So g_{θ} is finite on B . To see that g_{θ} is positive on B , we simply note that the Möbius transformation $1/(1-z)$ maps D to a set whose real part is greater than $1/2$.

□

Lemma 58. *The set E in the proof of Theorem 56 is residual in S .*

Proof. We use the same notation as in the proof of Theorem 56.

To show that

$$E = \bigcap_{t \in \mathbb{N}} \bigcup_{\theta \in P_t} C_{\theta}$$

is residual in S , it suffices to show that for each $t \in \mathbb{N}$,

$$S \setminus \bigcup_{\theta \in P_t} C_{\theta}$$

is nowhere dense in S . But, by Claim 1 (below), we have

$$S \setminus \bigcup_{\theta \in P_t} C_\theta \subset \bigcup_{\theta \in P_t^c} C_\theta.$$

(Here, we are defining $P_t^c := \{\theta \in [0, \pi)^n : \theta \notin P_t\}$. That is, we are taking the complement of P_t with respect to $[0, \pi)^n$.) So it suffices to show that, for each $t \in \mathbb{N}$,

$$\bigcup_{\theta \in P_t^c} C_\theta$$

is nowhere dense in S . Suppose not. Then there is a $t_0 \in \mathbb{N}$ such that $\bigcup_{\theta \in P_{t_0}^c} C_\theta$ is dense in some open set $U \subset S$. Without loss of generality, we may assume that for each $j = 1, \dots, n$

$$U \cap \{(z_1, \dots, z_{j-1}, x_j + 0i, z_{j+1}, \dots, z_n)\} = \emptyset.$$

In other words, we assume that each coordinate of each point in U has non-zero imaginary part.

Define $G := \{\theta \in [0, \pi)^n : C_\theta \cap U \neq \emptyset\}$. We show in Claim 2 (below) that G is open in $[0, \pi)^n$. We now show that $P_{t_0}^c$ is dense in G . Suppose not. Then $\exists w \in G$ and $\epsilon > 0$ such that $B(w, \epsilon) \subset G$ and such that

$$B(w, \epsilon) \cap P_{t_0}^c = \emptyset.$$

But in Claim 3 (below) we show that

$$U \cap \bigcup_{\theta \in B(w, \epsilon)} C_\theta$$

is open (and non-empty) in U . Hence, since $\bigcup_{\theta \in P_{t_0}^c} C_\theta$ is dense in U , we have

$$\exists p \in U \cap \left(\bigcup_{\theta \in B(w, \epsilon)} C_\theta \right) \cap \left(\bigcup_{\theta \in P_{t_0}^c} C_\theta \right).$$

So $\exists \phi \in P_{t_0}^c$ and $\exists \psi \in B(w, \epsilon)$ such that $U \cap C_\phi \cap C_\psi \neq \emptyset$. Thus, by Claim 4, $\phi = \psi$.

So

$$B(w, \epsilon) \cap P_{t_0}^c \neq \emptyset,$$

a contradiction of our earlier assertion that this intersection is empty. Thus $P_{t_0}^c$ is indeed dense in G . But, since G is open in $[0, \pi)^n$, this contradicts the fact that (by the way it was constructed) $P_{t_0}^c$ is nowhere dense in $[0, \pi)^n$. Thus, for each $t \in \mathbb{N}$,

$$\bigcup_{\theta \in P_t^c} C_\theta$$

is nowhere dense in S . This is what we wished to prove.

□

Claim 1. $\bigcup_{\theta \in [0, \pi)^n} C_\theta = S$.

Proof. Suppose $z \in S$. Then $z = (r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})$ for some $\theta_1, \dots, \theta_n \in [0, 2\pi)$ and $r_1, \dots, r_n \in [0, 1]$ with $r_1^2 + \dots + r_n^2 = 1$. For $j = 1, \dots, n$, let

$$\theta'_j = \begin{cases} \theta_j & \text{if } \theta_j \in [0, \pi) \\ \theta_j - \pi & \text{if } \theta_j \in [\pi, 2\pi) \end{cases}$$

Then $\theta' = (\theta'_1, \dots, \theta'_n) \in [0, \pi)^n$. We will now show that $z \in C_{\theta'}$. To show this, it suffices to show that $\sum_{j=1}^n (e^{-i\theta'_j} z_j)^2 = 1$. We have

$$\begin{aligned} \sum_{j=1}^n (e^{-i\theta'_j} z_j)^2 &= \sum_{j=1}^n (e^{-i\theta'_j} r_j e^{i\theta_j})^2 \\ &= \sum_{j=1}^n r_j^2 \left(e^{2i(\theta_j - \theta'_j)} \right) \\ &= \sum_{j=1}^n r_j^2 \\ &= 1. \end{aligned}$$

The claim has thus been established. \square

Claim 2. Let U be an open subset of S satisfying the assumption that each coordinate of each point of U has non-zero imaginary part. Then the set $G := \{\theta \in [0, \pi)^n : C_\theta \cap U \neq \emptyset\}$ is open in $[0, \pi)^n$.

Proof. Suppose not. Then there is a point $\theta \in G$ and a sequence of points $\{\phi_k\}_{k=1}^\infty$ in $[0, \pi)^n \setminus G$ such that $\lim_{k \rightarrow \infty} \phi_k = \theta$. Thus $C_\theta \cap U \neq \emptyset$, but $C_{\phi_k} \cap U = \emptyset$ for each $k \in \mathbb{N}$.

Let $m \in C_\theta \cap U$. We may write $m = (r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})$ where $\theta = (\theta_1, \dots, \theta_n)$ and $r_1^2 + \dots + r_n^2 = 1$. Since U is open, there is an $\epsilon > 0$ such that $S \cap B(m, \epsilon) \subset U$. Let $k \in \mathbb{N}$ be such that $\|\phi_k - \theta\| < \epsilon$. Let $w := (r_1 e^{i\phi_{k1}}, \dots, r_n e^{i\phi_{kn}})$ where $\phi_k = (\phi_{k1}, \dots, \phi_{kn})$. Note that $w \in C_{\phi_k}$. Then, making use of the fact that $|e^{ia} - e^{ib}| \leq |a - b|$ for every $a, b \in \mathbb{R}$, we have

$$\begin{aligned} \|m - w\|_{\mathbb{C}^n} &= \left(r_1^2 |e^{i\theta_1} - e^{i\phi_{k1}}|^2 + \dots + r_n^2 |e^{i\theta_n} - e^{i\phi_{kn}}|^2 \right)^{\frac{1}{2}} \\ &\leq \left(r_1^2 |\theta_1 - \phi_{k1}|^2 + \dots + r_n^2 |\theta_n - \phi_{kn}|^2 \right)^{\frac{1}{2}} \\ &\leq \left(|\theta_1 - \phi_{k1}|^2 + \dots + |\theta_n - \phi_{kn}|^2 \right)^{\frac{1}{2}} \\ &= \|\theta - \phi_k\|_{\mathbb{R}^n} \\ &< \epsilon. \end{aligned}$$

Thus $w \in S \cap B(m, \epsilon) \subset U$. But recall also that $w \in C_{\phi_k}$. Thus $w \in U \cap C_{\phi_k}$, contradicting the assumption that $C_{\phi_k} \cap U = \emptyset$ for each $k \in \mathbb{N}$. \square

Claim 3. Let U be an open subset of S satisfying the assumption that each

coordinate of each point of U has non-zero imaginary part. Given an open ball $B(w, \epsilon) \subset G := \{\theta \in [0, \pi)^n : C_\theta \cap U \neq \emptyset\}$, the set

$$A := U \cap \bigcup_{\theta \in B(w, \epsilon)} C_\theta$$

is open (and non-empty) in U .

Proof. It is clear that $A \neq \emptyset$. To show that A is open, we will show that for fixed $m \in A$, $\exists \epsilon''' > 0$ such that $(B(m, \epsilon''') \cap S) \subset A$. Let $m \in A$. Write $m = (r_1 e^{i\phi_1}, \dots, r_n e^{i\phi_n})$ where $\phi_j \in (0, \pi)$ for each $j = 1, \dots, n$ and where $r_1^2 + \dots + r_n^2 = 1$ with none of the r_j 's equal to zero (this assures that m is written *uniquely* as such). Note that $\phi = (\phi_1, \dots, \phi_n)$ is in $B(w, \epsilon)$. Let $\epsilon' > 0$ be given such that $B(\phi, \epsilon') \subset B(w, \epsilon)$. Note that since $\epsilon \in (0, \pi/2)$, we also have that $\epsilon' \in (0, \pi/2)$. Let

$$\epsilon'' = \min_{j \in \{1, \dots, n\}} r_j \sin(\epsilon' / \sqrt{n}),$$

and let

$$\begin{aligned} \epsilon''' &= \min\{\epsilon'', \text{dist}(m, S \setminus U), \text{Im}(r_1 e^{i\phi_1}), \dots, \text{Im}(r_n e^{i\phi_n})\} \\ &= \min\{\epsilon'', \text{dist}(m, S \setminus U), r_1 \sin(\phi_1), \dots, r_n \sin(\phi_n)\}. \end{aligned}$$

Suppose $y \in S$ such that $\|m - y\|_{\mathbb{C}^n} < \epsilon'''$ (i.e., $y \in B(m, \epsilon''')$). Write $y = (t_1 e^{i\psi_1}, \dots, t_n e^{i\psi_n})$ where $\psi_j \in [0, \pi)$ for each $j = 1, \dots, n$ and where $t_1^2 + \dots + t_n^2 = 1$. Since $\|m - y\|_{\mathbb{C}^n} < \epsilon'''$, then $\|m - y\|_{\mathbb{C}^n} < \epsilon''$. Thus, for each $j = 1, \dots, n$,

$$|r_j e^{i\phi_j} - t_j e^{i\psi_j}| \leq \|m - y\|_{\mathbb{C}^n} < \epsilon''' \leq \epsilon'' \leq r_j \sin(\epsilon' / \sqrt{n}).$$

But also, since $\epsilon''' \leq r_j \sin(\phi_j)$ for each fixed j , we have

$$|r_j e^{i\phi_j} - t_j e^{i\psi_j}| \leq \|m - y\|_{\mathbb{C}^n} < \epsilon''' \leq r_j \sin(\phi_j).$$

So the point $t_j e^{i\psi_j}$ is a point in the disc centered at $r_j e^{i\phi_j}$ with radius

$$r_j \cdot \min\{\sin(\epsilon'/\sqrt{n}), \sin(\phi_j)\}.$$

We claim that this implies that

$$|\phi_j - \psi_j| < \frac{\epsilon'}{\sqrt{n}}.$$

To see this, it suffices to show that if $se^{i\omega}$ (where $\omega \in [0, \pi)$) is *any* point in the closure of this disc, then

$$|\phi_j - \omega| < \frac{\epsilon'}{\sqrt{n}}.$$

So suppose $se^{i\omega}$ is a point in this closed disc that maximizes $|\phi_j - \omega|$. (There are, of course, two such points.) Let A be the origin, B be the point $se^{i\omega}$, and C be the point $r_j e^{i\phi_j}$. Then the angle at B in the triangle ABC is a right angle. The angle at A is $|\phi_j - \omega|$. (Here we have implicitly used our assumption that $|r_j e^{i\phi_j} - se^{i\omega}| \leq r_j \sin(\phi_j)$. That is, in order to avoid the problems with the argument function that occurs on the real axis, we have assumed that $r_j e^{i\phi_j}$ and $se^{i\omega}$ either both lie in the upper half-plane or both lie in the lower half-plane.) So $\sin(|\phi_j - \omega|) = \min\{\sin(\epsilon'/\sqrt{n}), \sin(\phi_j)\} \leq \sin(\epsilon'/\sqrt{n})$. But since $\epsilon'/\sqrt{n} \in (0, \pi/2)$ and since the sin function is increasing on $(0, \pi/2)$, we have that

$$|\phi_j - \omega| < \frac{\epsilon'}{\sqrt{n}}.$$

We have thus shown that

$$|\phi_j - \psi_j| < \frac{\epsilon'}{\sqrt{n}}$$

for each $j = 1, \dots, n$. Hence

$$\begin{aligned} \|\phi - \psi\|_{\mathbb{R}^n} &= (|\phi_1 - \psi_1|^2 + \dots + |\phi_n - \psi_n|^2)^{1/2} \\ &< \left(\frac{n\epsilon'^2}{n}\right)^{1/2} \\ &= \epsilon'. \end{aligned}$$

So $\psi \in B(\phi, \epsilon') \subset B(w, \epsilon)$. But $y \in C_\psi$, so $y \in \bigcup_{\theta \in B(w, \epsilon)} C_\theta$. Also, $y \in U$ (since $\|m - y\|_{\mathbb{C}^n} < \epsilon''' \leq \text{dist}(m, S \setminus U)$). Therefore, $y \in U \cap \bigcup_{\theta \in B(w, \epsilon)} C_\theta = A$. Thus, A is open in U . \square

Claim 4. Given the set $U \subset S$ (satisfying the assumption that each coordinate of each point of U has non-zero imaginary part) and given $\phi, \psi \in [0, \pi)^n$ such that $U \cap C_\phi \cap C_\psi \neq \emptyset$, then $\phi = \psi$.

Proof. Suppose $U \cap C_\phi \cap C_\psi \neq \emptyset$, and let $z \in U \cap C_\phi \cap C_\psi$. Write $z = (z_1, \dots, z_n) = (r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})$ where $\theta_j \in (0, \pi)$ and $r_j \in (-1, 1)$. None of the r_j 's is zero, by the assumption on U . Since $z \in C_\phi$, then

$$1 = \sum_{j=1}^n (e^{-i\phi_j} z_j)^2.$$

So

$$1 = \sum_{j=1}^n r_j^2 e^{2i(\theta_j - \phi_j)}.$$

But since $\sum_{j=1}^n r_j^2 = 1$ with none of the r_j 's equal to zero, and since $\theta_j, \phi_j \in [0, \pi)$, we have that $\theta_j = \phi_j$ for each $j = 1, \dots, n$. Similarly, $\theta_j = \psi_j$ for each $j = 1, \dots, n$. Thus $\phi = \psi$. \square

CHAPTER V

OTHER PROPERTIES OF RADIAL LIMIT ZERO SETS

In this chapter, we will prove several miscellaneous results related to radial limit zero sets.

A. Dependence on Complex Structure

It will follow from Proposition 59 that the “complex structure” on S must take some role in any characterization of radial limit zero sets of holomorphic functions on B .

Before we state and prove Proposition 59, we first make some preliminary definitions and remarks. Identify \mathbb{C}^2 with \mathbb{R}^4 via the map $(a + bi, c + di) \mapsto (a, b, c, d)$. For $k \in \mathbb{N}$, define

$$L_k := \{(se^{i/k}, te^{i/k}) \in \mathbb{C}^2 : s, t \in \mathbb{R}\}$$

and

$$L_k^* := \{((s + it) \cos(1/k), (s + it) \sin(1/k)) \in \mathbb{C}^2 : s, t \in \mathbb{R}\}.$$

Also define $L_0 := \{(s, t) \in \mathbb{C}^2 : s, t \in \mathbb{R}\}$ and $L_0^* := \{(s + it, 0) \in \mathbb{C}^2 : s, t \in \mathbb{R}\}$.

It is evident that the L_k^* 's are complex lines, but the L_k 's are not complex lines. Note also that the L_k 's are unitarily equivalent (that is, given any two L_k 's, there is a unitary transformation mapping one onto the other). To see this, it suffices to show that L_0 is unitarily equivalent to a given L_k . The unitary transformation $\mathcal{U}(z_1, z_2) = (e^{i/k}z_1, e^{i/k}z_2)$ maps L_0 onto L_k . The L_k^* 's are also unitarily equivalent (since they are all complex lines).

Proposition 59. *Let $n \geq 2$. There is a set of points $A_1 \subset S$, an orthogonal transformation \mathcal{O} of the underlying real coordinates \mathbb{R}^{2n} , and a nonconstant holomorphic*

function $f : B \rightarrow \mathbb{C}$ having radial limit zero on A_1 such that there does not exist a nonconstant holomorphic function on B having radial limit zero on $A_2 := \mathcal{O}(A_1)$.

Proof. The proposition will be proved for $n = 2$.

The orthogonal transformation $\mathcal{O} : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ defined by $\mathcal{O}(a, b, c, d) = (a, c, b, d)$ maps L_k onto L_k^* (and, similarly, it maps L_k^* onto L_k). Note that \mathcal{O} is not a unitary transformation (when considered as a mapping from \mathbb{C}^2 to \mathbb{C}^2).

Define subsets A_1 and A_2 of S as follows:

$$A_1 := \left(\bigcup_{k=0}^{\infty} L_k \right) \cap S$$

and

$$A_2 := \left(\bigcup_{k=0}^{\infty} L_k^* \right) \cap S.$$

Note that \mathcal{O} maps A_1 onto A_2 .

We will show that there exists a nonconstant holomorphic function $f : B \rightarrow \mathbb{C}$ with radial limit zero on A_1 , but there does not exist a nonconstant holomorphic function with radial limit zero on A_2 .

Define $g_0(z_1, z_2) = 1 - (z_1^2 + z_2^2)$. It is not hard to see that g_0 vanishes on the circle $L_0 \cap S$ and that it does not vanish anywhere else on \overline{B} . So the function $\frac{1}{g_0}$ is holomorphic on B and continuous on $\overline{B} \setminus (L_0 \cap S)$. Also note that $\text{Im}(\frac{1}{g_0}) \equiv 0$ on L_0 .

Similarly, define $g_k(z_1, z_2) = 1 - ((e^{-i/k} z_1)^2 + (e^{-i/k} z_2)^2)$. Note that g_k vanishes on the circle $L_k \cap S$ and that it does not vanish anywhere else on \overline{B} . So $\frac{1}{g_k}$ is holomorphic on B and continuous on $\overline{B} \setminus (L_k \cap S)$. Also, $\text{Im}(\frac{1}{g_k}) \equiv 0$ on L_k .

For $k \in \mathbb{N}$, let

$$M_k = 2^k \max \left\{ \left| \frac{1}{g_k(z)} \right| : z \in \overline{B}(0, 1 - 1/k) \cup \bigcup_{j=0}^{k-1} (\overline{B} \cap L_j) \right\},$$

and set $M_0 = 1$. Note that $\frac{1}{M_k} \operatorname{Re}(\frac{1}{g_k})$ is a harmonic function on B that is “close” to 0 on $B(0, 1 - 1/k) \cup L_0 \cup L_1 \cup \dots \cup L_{k-1}$ and that blows up to $+\infty$ on L_k . Now, for $k \in \mathbb{N}$, define

$$f_k(z) := e^{-\frac{1}{M_k \cdot g_k(z)}}.$$

Thus f_k is a holomorphic function on B that is “close” to 1 on $B(0, 1 - 1/k) \cup L_0 \cup L_1 \cup \dots \cup L_{k-1}$ and has radial limit zero on $L_k \cap S$. Define

$$f(z) = \prod_{k=0}^{\infty} f_k(z).$$

This product converges (and is not identically zero) on B , and f has radial limit zero on A_1 .

We now show that if a holomorphic function $h : B \rightarrow \mathbb{C}$ has radial limit zero on A_2 , then $h \equiv 0$. By the Lusin-Privalov Theorem (in dimension one), h must vanish on each L_k^* ($k = 0, 1, 2, \dots$). To avoid fractions, we will re-scale and assume the function is defined on $B(0, 3)$.

Consider the real line $\gamma_0 := \{(1, 1, t, t) \in \mathbb{R}^4 : t \in \mathbb{R}\}$. Each of the L_k^* ’s intersects γ_0 , and the points of intersection have a limit point on $\gamma_0 \cap B$. So h is identically zero on the intersection of $B(0, 3)$ with the complex line determined by γ_0 . This complex line is $\Gamma_0 := \{(1, 1, s - t, s + t) : s, t \in \mathbb{R}\} = \{(1 + i, z) : z \in \mathbb{C}\}$. Similarly, for each $j \in \mathbb{N}$, we have that h is identically zero on the intersection of $B(0, 3)$ with the complex line $\Gamma_j := \{(\frac{j}{j+1}, \frac{j}{j+1}, s - t, s + t) : s, t \in \mathbb{R}\} = \{\frac{j}{j+1}(1 + i, z) : z \in \mathbb{C}\}$. We will show that this implies that $h \equiv 0$ on $B(0, 3)$.

Fix a point $(z_1, z_2) \in B(0, 3)$. The real line $\{(r(1+i), z_2) : r \in \mathbb{R}\}$ intersects Γ_j for $j \in \mathbb{N} \cup \{0\}$. The complex line determined by this real line is $\{(r(1+i) + p(i-1), z_2) : r, p \in \mathbb{R}\} = \{(z, z_2) : z \in \mathbb{C}\}$. So h must be identically zero on this complex line. But

since this complex line contains the point (z_1, z_2) , we have $h(z_1, z_2) = 0$. Thus $h \equiv 0$ on $B(0, 3)$.

□

B. Not Closed Under Unions

In dimension one, it is clear that radial limit zero sets of holomorphic functions on D are not closed under unions (even finite unions). For example, let E be a first category subset of T having full measure in T . Then E and E^c are each radial limit zero sets (by Corollary 19 and Lemma 49, respectively), but the only holomorphic function having radial limit zero on $T = E \cup E^c$ is the identically zero function.

Since neither Corollary 19 nor Lemma 49 is true in higher dimensions, the question still remains whether radial limit zero sets in higher dimensions are closed under finite unions. We will show that they are not.

Proposition 60. *There exist subsets $E_1, E_2 \subset S$ with the following property: there are nonconstant holomorphic functions $f, g : B \rightarrow \mathbb{C}$ having radial limit zero on E_1 and E_2 , respectively, but there is no nonconstant holomorphic function on B with radial limit zero on $E_1 \cup E_2$.*

Proof. Let $\phi : D \rightarrow \mathbb{C}$ be a holomorphic function whose real part has radial limit $+\infty$ on a set $H \subset T$ that is both residual and measure zero in T . This is possible by the proof of Lemma 49. Also, let $\psi : D \rightarrow \mathbb{C}$ be a holomorphic function whose real part has radial limit $+\infty$ on $T \setminus H$. This is possible by Lemma 51.

For $k \in \mathbb{N} \cup \{0\}$, let \mathcal{U}_k be a unitary transformation that maps L_k^* onto L_0^* . Write $\mathcal{U}_k = (\mathcal{U}_{k,1}, \mathcal{U}_{k,2})$. Note that $\mathcal{U}_{k,1}$ maps B to D . Define functions $\phi_k : B \rightarrow \mathbb{C}$ and $\psi_k : B \rightarrow \mathbb{C}$ by

$$\phi_k(z_1, z_2) = \phi(\mathcal{U}_{k,1}(z_1, z_2))$$

and

$$\psi_k(z_1, z_2) = \psi(\mathcal{U}_{k,1}(z_1, z_2)).$$

Note that ϕ_k and ψ_k are continuous up to $\overline{B} \setminus (L_k^* \cap S)$, and the union of the sets on which their real parts have radial limit $+\infty$ is $L_k^* \cap S$.

For $k \in \mathbb{N}$, let

$$M_k = 2^k \cdot \max\{|\phi_k(z)| : z \in B(0, 1 - 1/k) \cup \bigcup_{j=0}^{k-1} (\overline{B} \cap L_j^*)\}$$

and

$$N_k = 2^k \cdot \max\{|\psi_k(z)| : z \in B(0, 1 - 1/k) \cup \bigcup_{j=0}^{k-1} (\overline{B} \cap L_j^*)\}.$$

Finally, define holomorphic functions f and g on B by

$$f(z) = \prod_{k=0}^{\infty} e^{-\frac{\phi_k(z)}{M_k}}$$

and

$$g(z) = \prod_{k=0}^{\infty} e^{-\frac{\psi_k(z)}{M_k}}.$$

Note that these products converge uniformly on compact subsets of B . Also, the union of the sets on which f and g have radial limit zero (call them E_1 and E_2 , respectively) is the set A_2 from the proof of Proposition 59. So, as before, any holomorphic function having radial limit zero on $A_2 = E_1 \cup E_2$ must be identically zero.

□

C. Smooth Curves of Finite Length in S that are Non-tangential Uniqueness Sets

In this section, we will show that there are “small” boundary uniqueness sets in S . To avoid confusion, we mention that the following example (from [35, pp. 222-223]) does *not* deal with “radial limit” uniqueness but a different kind of uniqueness.

Example:

A set $K \subset S$ is said to be a *determining set* if any holomorphic function $f : B \rightarrow \mathbb{C}$ which is continuous on \overline{B} and which has $f(\zeta) = 0$ for all $\zeta \in K$ must be identically zero. Suppose $\alpha_1, \dots, \alpha_n$ are positive real numbers that are linearly independent over the rationals. Define $F : \mathbb{C} \rightarrow \mathbb{C}^n$ by

$$F(z) = \frac{1}{\sqrt{n}}(e^{i\alpha_1 z}, \dots, e^{i\alpha_n z}).$$

If I is an interval on the real axis, we claim that $F(I)$ is a determining set. Notice that $F(I)$ is a “small” set (i.e., it is a 1-(real)-dimensional curve). Suppose f is holomorphic on B and continuous on \overline{B} with $f|_{F(I)} = 0$. Note that F maps the upper half-plane into B . So $f \circ F$ is a holomorphic function on the (open) upper half-plane that is continuous on the closed upper half-plane and is 0 on I . Thus $f \circ F \equiv 0$ on the closed upper half-plane. So, if R denotes the real axis, then $f(\zeta) = 0$ for every $\zeta \in F(R)$. But $F(R)$ is dense in the torus $\{z : |z_1| = \dots = |z_n| = n^{-1/2}\}$. (This follows from our assumption about the α_j ’s.) Thus, by continuity, $f \equiv 0$ on the whole torus. By the Cauchy formula in \mathbb{C}^n , $f \equiv 0$ on B .

The question remains as to whether the set $F(I)$ from the above example is a radial or non-tangential uniqueness set. We give a partial answer to this question. We have not done much new here other than make a few simple observations.

Theorem 61. *If a holomorphic function $f : B \rightarrow \mathbb{C}$ has non-tangential limit zero on the set $F(I)$ from the above example, then $f \equiv 0$.*

We will see from the proof that we can weaken the hypotheses of the above theorem considerably. For example, we do not need to assume that the full non-tangential limit exists, but only that the radial limit exists and that it is bounded in

each of the regions $D_t(\omega)$ where $t > 1$ and $\omega \in F(I)$ (see [35, p. 174]). Also, we do not need to assume that this limit exists along all of $F(I)$ but only along a subset that is both metrically dense and second category in $F(I)$. Also, by assuming the function is defined on D^n instead of B , we can get an improvement of part 2 of a theorem of M. Tsuji (Theorem 41). To see this, we note that any function having non-tangential limit zero (in the sense that the limit is zero as each z_k approaches $|z_k| = 1$ non-tangentially) on a positive measure subset of the torus T^n must have positive measure on some curve of the same form as $F(I)$.

Proof. Let $f : B \rightarrow \mathbb{C}$ be a holomorphic function with non-tangential limit zero on $F(I)$. Fix a point $x_0 \in I$. We will show that $\lim_{t \rightarrow 0^+} f(F(x_0 + it)) = 0$. Since f has non-tangential limit zero at $F(x_0)$, it suffices to show that the curve $F(x_0 + it)$ approaches $F(x_0)$ non-tangentially as $t \rightarrow 0^+$. So we must show that the vectors

$$\frac{1}{\sqrt{n}}(e^{i\alpha_1 x_0}, \dots, e^{i\alpha_n x_0})$$

and

$$\frac{d}{dt}F(x_0 + it)|_{t=0}$$

are real orthogonal. We compute:

$$\begin{aligned} \frac{d}{dt}F(x_0 + it)|_{t=0} &= \frac{1}{\sqrt{n}} \frac{d}{dt} (e^{i\alpha_1(x_0+it)}, \dots, e^{i\alpha_n(x_0+it)})|_{t=0} \\ &= \frac{-1}{\sqrt{n}} (\alpha_1 e^{i\alpha_1 x_0}, \dots, \alpha_n e^{i\alpha_n x_0}). \end{aligned}$$

The real inner product of the two vectors is then easily computed to be

$$\frac{-1}{n} (\alpha_1 + \dots + \alpha_n),$$

which is non-zero by the original assumption about the α_j 's.

Since x_0 was arbitrary in the interval I , we have (by an easy generalization of

the Lusin-Privalov theorem to the upper half-plane) that $f \circ F$ is identically zero on the upper half-plane. In particular, $f(F(x+i))$ is zero for each $x \in \mathbb{R}$. But the curve $\{F(x+i) : x \in \mathbb{R}\}$ is dense in the torus $|z_1| = \frac{1}{e^{\alpha_1 \cdot \sqrt{n}}}, \dots, |z_n| = \frac{1}{e^{\alpha_n \cdot \sqrt{n}}}$, which is contained entirely in B . By the continuity of f , the Cauchy formula, and the identity theorem, we have that $f \equiv 0$ on B .

□

CHAPTER VI

CONCLUSION

A. Summary of Results

In this dissertation, we have proved a higher dimensional version of the classical Lusin-Privalov Radial Uniqueness Theorem, to the effect that radial limit zero sets of nonconstant holomorphic functions on B must be locally “small” in a sense that involves both Lebesgue measure and Baire category. The original proof that Lusin and Privalov gave does not immediately extend to higher dimensions because it uses the Riemann mapping theorem. However, after making a few modifications, we obtained a proof using similar methods. We have also shown that, in contrast to what is the case in dimension one, the converse is not true in higher dimensions.

Privalov showed how to construct holomorphic functions on D that have *residual* radial limit zero sets. However, his method of construction does not generalize to higher dimensions because it uses the fact that harmonic functions are locally real parts of holomorphic functions. We have given a new construction in dimension one of a holomorphic function having radial limit zero on a residual subset of T . We have also shown how this construction may be generalized to higher dimensions to obtain a holomorphic function on B with radial limit zero on a residual subset of S .

We have shown that there is a set $E \subset S$, an orthogonal transformation \mathcal{O} of the underlying real coordinates of \mathbb{C}^n , and a holomorphic function having radial limit zero on E such that there is no nonconstant holomorphic function having radial limit zero on $\mathcal{O}(E)$. Thus, we have shown that any characterization of radial limit zero sets of nonconstant holomorphic functions on B must take into account the “complex structure” on S . We have also shown that, as is the case in dimension one, the class of

radial limit zero sets of nonconstant holomorphic functions is not closed under finite unions.

Finally, we have shown that there are smooth curves in S of finite length that are non-tangential limit uniqueness sets for holomorphic functions on B . To do this, we have merely made a few simple observations about certain curves that are known to be uniqueness sets for holomorphic functions on B that have continuous extensions to S .

B. Open Questions

There remain several open problems, the most important of which is to find a *characterization* of the radial limit zero sets of nonconstant holomorphic functions on B . We have already shown a few things that must be true about any such a characterization. For example, it must allow for residual sets and full measure sets, but it cannot allow for all sets that are simultaneously measure zero and first category in S . Any such characterization will also have to take into account the complex structure on S .

One thing that is not known is whether there exists a set $E \subset S$ ($n \geq 2$) with the property that both E and E^c are radial limit zero sets for nonconstant holomorphic functions. In dimension one, it is known that such sets do indeed exist. For example, just take E to be a first category subset of T having full measure and apply the Bagemihl-Siedel theorem in an appropriate way.

It is not hard to construct, using Runge's Theorem, a holomorphic function on D such that the set of points of T for which the radial limit fails to exist is all of T . Since Runge's Theorem does not hold in higher dimensions, it is not known whether such a function exists in higher dimensions (where, of course, D and T are replaced by B and S , respectively).

Another open problem is to give a characterization of the subsets of S for which a Bagemihl-Siedel type of theorem holds. In one dimension, it is not hard to show that such an approximation can be done if and only if the set is first category in T . In higher dimensions, we know by the result of Hakim and Sibony that there exist full measure subsets of S on which such approximation is possible. In fact, it follows from Dupain's result that there are subsets of S having full linear measure in every great circle on which such approximation is possible. (Recall that a set $E \subset S$ is said to be a *great circle* if it is the intersection of S with some complex line containing the origin.) In Dupain's result, the set is, in fact, first category on each great circle. It is not known whether the type of approximation under consideration is possible on *every* set that is first category on each great circle.

REFERENCES

- [1] M. G. Arsove, *The Lusin-Privalov theorem for subharmonic functions*, Proc. London Math. Soc. **14** (3) (1964), 260-270.
- [2] S. Axler, P. Bourdon, W. Ramey, *Harmonic Function Theory*, 2nd ed., Springer-Verlag, New York, 2001.
- [3] F. Bagemihl, W. Seidel, *Some boundary properties of analytic functions*, Math. Z. **61** (1954), 186-199.
- [4] K. F. Barth, W. J. Schneider, *An asymptotic analogue of the F. and M. Riesz radial uniqueness theorem*, Proc. Amer. Math. Soc. **22** (1969), 53-54.
- [5] F. Bayart, *Universal radial limits of holomorphic functions*, Glasgow Math. J. **47** (2005), 261-267.
- [6] R. D. Berman, *Elementary proofs of some asymptotic radial uniqueness theorems*, Proc. Amer. Math. Soc. **86** (1982), 226-229.
- [7] R. D. Berman, *A converse to the Lusin-Privalov radial uniqueness theorem*, Proc. Amer. Math. Soc. **87** (1983), 103-106.
- [8] G. E. Bredon, *Topology and Geometry*, Springer-Verlag, New York, 1993.
- [9] E. M. Čirka, *The Lindelöf and Fatou theorems in \mathbb{C}^n* , Mat. Sb. **92** (1973), 622-644.
- [10] E. F. Collingwood, A. J. Lohwater, *The Theory of Cluster Sets*, Cambridge Univ. Press, Cambridge, 1966.

- [11] J. B. Conway, *Functions of One Complex Variable II*, Springer-Verlag, New York, 1995.
- [12] Y. Dupain, *Limites radiales de fonctions holomorphes dans la boule unite de \mathbb{C}^n* , J. D'Analyse Math. **52** (1989), 107-116.
- [13] P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math. **30** (1906), 335-400.
- [14] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., John Wiley and Sons, New York, 1999.
- [15] S. J. Gardiner, *The Lusin-Privalov theorem for subharmonic functions*, Proc. Amer. Math. Soc. **124** (1996) 3721-3727.
- [16] S. J. Gardiner, W. Hansen, *Boundary sets where harmonic functions may become infinite*, Math. Ann. **323** (2002), 41-54.
- [17] M. Hakim, N. Sibony, *Boundary properties of analytic functions in the ball of \mathbb{C}^n* , Math. Ann. **276** (1987), 549-555.
- [18] T. J. Kaczynski, *Boundary functions for bounded harmonic functions*, Trans. Amer. Math. Soc. **137** (1969), 203-209.
- [19] J. P. Kahane, Y. Katznelson, *Sur le comportement radial des fonctions analytiques*, C. R. Acad. Sci. Paris **272** (1971), A718-A719.
- [20] S. V. Kolesnikov, *On the sets of nonexistence of radial limits of bounded analytic functions*, Russian Acad. Sci. Sb. Math. **81** (1995), 477-485.
- [21] A. Korányi, *Harmonic functions on Hermitian hyperbolic space*, Trans. Amer. Math. Soc. **135** (1969), 507-516.

- [22] S. G. Krantz, *Function Theory of Several Complex Variables*, 2nd ed., Amer. Math. Soc., Providence, RI, 1992.
- [23] S. G. Krantz, *Geometric Function Theory: Explorations in Complex Analysis*, Birkhäuser, Boston, MA, 2006.
- [24] S. G. Krantz, *The Lindelöf principle in several complex variables*, J. Math. Anal. Appl. **326** (2007), 1190-1198.
- [25] E. Lindelöf, *Sur un principe générale de l'analyse et ses applications à la theorie de la représentation conforme*, Acta Soc. Sci. Fennicae **46** (1915), 1-35.
- [26] N. Lusin, I. Privalov, *Sur l'unicité et la multiplicité des fonctions analytiques*, Ann. Sci. École Norm. Sup. **42** (1925), 143-191.
- [27] J. E. McMillan, *On radial limits and uniqueness of holomorphic functions*, Math. Zeitschr. **92** (1966), 321-322.
- [28] A. Nagel, W. Rudin, *Local boundary behavior of bounded holomorphic functions*, Canad. J. Math. **30** (1978), 583-592.
- [29] I. I. Privalov, *Randeigenschaften Analytischer Funktionen*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1956.
- [30] F. Riesz, M. Riesz, *Über die Randwerte einer analytischen Funktion*, Quatrième Congrès des Math. Scand. Stockholm (1916), 27-44.
- [31] W. Rudin, *Function Theory in the Unit Ball of \mathbb{C}^n* , Grundlehren Math. Wiss., vol. 241, Springer-Verlag, New York-Berlin, 1980.
- [32] W. Rudin, *New Constructions of Functions Holomorphic in the Unit Ball of \mathbb{C}^n* , American Mathematical Society, Providence, RI, 1986.

- [33] K. F. Tse, *An analogue of the Lusin-Privaloff radial uniqueness theorem*, Proc. Amer. Math. Soc. **25** (1970), 310-312.
- [34] M. Tsuji, *On the boundary value of a bounded analytic function of several complex variables*, Proc. Japan Acad. **21** (1949), 308-312.

VITA

Michael C. Fulkerson earned his Bachelor of Science in Education degree from Central Michigan University in May 2001. He entered the graduate program in mathematics at Central Michigan University, where he received his Master of Arts degree in December 2003. He then entered the doctoral program in mathematics at Texas A&M University, where he received his Ph.D. in August 2008.

Dr. Fulkerson may be reached at: Department of Mathematics, Texas A&M University, Mailstop 3368, College Station, TX 77843-3368. He may also be reached by email at: michaelf@math.tamu.edu.